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On Collatz theorem II

A new proof of the $3n + 1$ conjecture

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Abstract. We present a new, simpler proof of Collatz conjecture:
for every natural number $n > 0$ the computation of Collatz algorithm is finite.

Key words: algorithm of Collatz, halting property, Collatz tree, program calculus $\mathcal{AL}$.

Contents

1. Introduction

The $3x + 1$ problem remained open for over 80 years. It was formulated in 1937 by Lothar Collatz. The problem became quite popular due to its wording, for it is short and easy to comprehend.
Collatz remarked that for any given natural number $n > 0$, the sequence $\{n_i\}$ defined by the following recurrence

$$\left.\begin{array}{l} n_0 = n \\ n_{i+1} = \begin{cases} n_i \div 2 & \text{when } n_i \text{ is even} \\ 3 \cdot n_i + 1 & \text{when } n_i \text{ is odd} \end{cases} \quad \text{for } \ i \geq 0 \end{array}\right\} \tag{rec1}$$

seem always reach the value 1.

He formulated the following conjecture

$$\text{for all } n \text{ exists } i \text{ such that } n_i = 1 \tag{Collatz conjecture}$$

One can give another formulation of the hypothesis of Collatz [1].
The number of papers devoted to the problem surpasses 200, c.f. [Lag10] . It is worthwhile to consult social media: wikipedia, youtube etc, there you can find some surprising ideas to prove the Collatz hypothesis as well as a technical analysis of the problem.
Computers are used and still work in the search of an eventual counterexample to the Collatz conjecture. The reports on progress appear each year. We claim that the counterexample approach is pointless, i.e. the computers can be turned off. We shall prove that the program that searches a counterexample will never stop.
Our goal will be achieved if we prove that for each number n the computation of the following Cl algorithm is finite.

$$Cl : \left\{\begin{array}{l} \textbf{while } n \neq 1 \textbf{ do} \\ \quad \textbf{if } even(n) \textbf{ then } n := n \div 2 \textbf{ else } n := 3n + 1 \textbf{ fi} \\ \textbf{od} \end{array}\right\}$$

Now, a couple of questions arise. Programs (algorithms) are expressions without possibility to assign a logical meaning to them. We need a formula Θ_{Cl} such that it obviously expresses the termination property of program Cl, a verifiable proof Π of the formula and a definition of relation $\mathcal{C}$ of deducibility.

[1] Let $f(n,0) \stackrel{df}{=} n$, and $f(n,i+1) \stackrel{df}{=} \begin{cases} f(n,i)/2 & \text{if } f(n,i) \text{ is even} \\ 3 \cdot f(n,i) + 1 & \text{if } f(n,i) \text{ is odd} \end{cases}$. Now, conjecture reads $\forall_n \exists_i f(n,i) = 1$.

Ah, we need also a specification of the domain in which the algorithm is to be executed, i.e. the axioms $\mathcal{A}x$ of the algebraic structure of natural numbers.

QUESTION 1. How to express the termination property of a program K as a formula Θ_K (i.e. a Boolean expression)?

Note, there is no first-order logical formula that express the halting property. First, let us recall the theorem on incompleteness of arithmetics, cf. Kurt Gödel . According to Gödel, the property *to be a natural number* is not expressible by any set of first-order formulas. The reader may wish to note, that halting property of the algoritm

$$q := 0; \textbf{while } q \neq n \textbf{ do } q := q + 1 \textbf{ od}$$

is valid in a data structure iff n is a standard (i.e. reachable) natural number. Therefore the halting property allow to define the set of natural numbers. In this situation it seems natural to pass from first-order language to larger algorithmic language. We enlarge the set of well formed expressions: beside terms and formulas of first order language we accept algorithms and we modify the definition of logical formulas. The simplest algorithmic formulas are of the form: $\langle$*algorithm*$\rangle\langle$*formula*$\rangle$.
As an example of an algorithmic formula consider the expression

$$\forall_n \{q := 0; \textbf{while } q \neq n \textbf{ do } q := q + 1 \textbf{ od}\}\,(n = q)$$

The latter formula is valid iff every element n can be reached from 0 by adding 1.
Now our goal is to prove the following formula

$$\forall_n \left\{ \begin{array}{l} \textbf{while } n \neq 1 \textbf{ do} \\ \quad \textbf{if } even(n) \textbf{ then } n := n \div 2 \textbf{ else } n := 3n + 1 \textbf{ fi} \\ \textbf{od} \end{array} \right\} (n = 1) \tag{1}$$

from the axioms of algorithmic theory of natural numbers $\mathcal{ATN}$, c.f. subsection 7.4. For the formula (1) expresses the termination property of program Cl.

QUESTION 2. How to prove such algorithmic formula?
Note, all structures that assure the validity of axioms are isomorphic (this is the categoricity theorem). Therefore, the termination formula, can be either proved (with the inference rules and axioms of calculus of programs $\mathcal{AL}$, or validated in this unique model of axioms of $\mathcal{ATN}$.

Let us make made a simple observation. The computation of Collatz algorithm if succesful goes through intermediate values. The following diagram illustrates a computation where odd numbers were encountered x times.

$$n \to \frac{n}{2^{k_0}} \to \left(3 * (\frac{n}{2^{k_0}}) + 1\right) \to \left(3 * (\frac{n}{2^{k_0}}) + 1\right)/2^{k_1} \to \cdots \to \frac{(\cdots)}{2^{k_x}} = 1 \tag{Dg}$$

where $k_0 = exp(n, 2), k_1 = exp(3 \cdot \frac{n}{2^{k_0}} + 1, 2), k_2 = exp(3 * \frac{(3 \cdot \frac{n}{2^{k_0}} + 1)}{2^{k_1}} + 1, 2), \ldots$ [2].

[2] **Note**, the function exp returns the largest exponent of 2 in the prime factorization of number x .
$exp(x, 2) \stackrel{df}{=} \{l := 0; y := x; \textbf{while } even(y) \textbf{ do } l := l + 1;\ y := y/2 \textbf{ od}\}(\text{result} = l)$ i.e. $l = exp(x, 2)$.

How the present paper is related to our earlier one? In our previous paper [MS21] we remarked that the computation of Collatz algorithm is finite iff there exist three natural numbers x, y, z such that:
a) the equation $n \cdot 3^x + y = 2^z$ is satisfied and
b) the computation of another algorithm IC is finite, the algorithm computes on triples$\langle x, y, z\rangle$.
It is worthwhile to mention that the subsequent triples are decreasing.
The proof we wrote there [MS21] is overly complicated.

Here we show that the 4-argument relation

$$\{n, x, y, z\} : \{IC\}(true).$$

is elementary recursive, since it may be expressed by an arihmetic expression with operator $\sum$.
The present paper shows arguments simpler and easier to follow.

Main points of the proof

- Collatz algorithm does not require multiplication operation, addition suffices.
- There is a computable data structure $\mathfrak{M}$ (c.f. table 1, page 22) such that:
  a) there are infinite computations of Collatz algorithm (c.f. subsection 7.1) and
  b) all axioms of *elementary theory of addition in natural numbers* (c.f. subsection 7.2) are valid in the structure $\mathfrak{M}$.
- Therefore the Collatz hypothesis is not a theorem of the first-order theory of addition, nor Peano arithmetic.
- Moreover, the hypothesis can not be written as a first-order formula.
- For a given element n Collatz algorithm terminates iff there exist three elements x, y, z such that the equality $n \cdot 3^x + y = 2^z$ holds and triple's algorithm IC terminates (c.f. [MS21]).
- The second,underlined part of the latter conjunction can be stated as: *there is the least triple* x, y, z *that satisfies following conjunction*

$$n \cdot 3^x + \left(\sum_{j=0}^{x-1}\left(3^{x-1-j} \cdot 2^{\sum_{j=0}^{x_f} k_j}\right)\right) = 2^{\sum_{j=0}^{x} k_j} \ \wedge \ z = \sum_{j=0}^{x} k_j \ \wedge \ y = 2^z - n \cdot 3^x$$

- From this we will show that, for every standard, reachable natural number n, Collatz algorithm halts, or equivalently, if for an element n the computation of Collatz algorithm is infinite, then the element n is not a standard natural number (see below).

2. A counterexample

We argue, that the formulation of the Collatz problem must be made with more precision. For there are several algebraic structures that can be viewed as structure of natural numbers of addition. Some of them admit infinite computations of Collatz algorithm.
We recall less known fact: arithmetic (i.e. first-order theory of natural numbers) has standard (*Archimedean*)

model $\mathfrak{N}$ as well as another *non-Archimedean* model $\mathfrak{M}$[3]. The latter structure allows for the existence of infinitely great elements.
Goedel's incompleteness theorem shows that there is no elementary theory T of natural numbers, such that every model is isomorphic to the standard model.
Two things are missing from the commonly accepted text: 1) What do we mean by proof? 2) what properties of natural numbers can be used in the proof? We recall an algebraic structure $\mathfrak{M}$ that models [Grz71] all axioms of elementary theory of addition of natural numbers, yet it admits unreachable elements [Tar34]. It means that the model contains element ε, such that the computation of Collatz algorithm that starts with ε is infinite.

Example of a finite execution

$$\langle 13,0\rangle \overset{\times 3+1}{\to} \langle 40,0\rangle \overset{\div 2}{\to} \langle 20,0\rangle \overset{\div 2}{\to} \langle 10,0\rangle \overset{\div 2}{\to}, \langle 5,0\rangle \overset{\times 3+1}{\to} \langle 16,0\rangle \overset{\div 2}{\to} \langle 8,0\rangle \overset{\div 2}{\to} \langle 4,0\rangle \overset{\div 2}{\to} \langle 2,0\rangle \overset{\div 2}{\to} \langle 1,0\rangle$$

Example of an infinite execution

$$\langle 8,\frac{1}{2}\rangle \overset{\div 2}{\to} \langle 4,\frac{1}{4}\rangle \overset{\div 2}{\to} \langle 2,\frac{1}{8}\rangle \overset{\div 2}{\to} \langle 1,\frac{1}{16}\rangle \overset{\times 3+1}{\to} \langle 4,\frac{3}{16}\rangle \overset{\div 2}{\to} \langle 2,\frac{3}{32}\rangle \overset{\div 2}{\to} \langle 1,\frac{3}{64}\rangle, \overset{\times 3+1}{\to} \langle 4,\frac{9}{64}\rangle \overset{\div 2}{\to} \langle 2,\frac{9}{128}\rangle \overset{\div 2}{\to} \dots$$

As you can guess, the data structure contains pairs $\langle k, w\rangle$ where k is an integer and w is a non-negative, rational number. The addition operation is defined componentwise. A pair $\langle k, w\rangle$ divided by 2 returns $\langle k \div 2, w \div 2\rangle$.
The reader may prefer to think of complex numbers instead of pairs, e.g. $(2 + \frac{9}{128}i)$ may replace the pair $\langle 2, \frac{9}{128}\rangle$.
The following observation seems to be of importance:.

Remark 2.1. *There exists an infinite computation* **c** *of Collatz algorithm in the structure* $\mathfrak{M}$ *, such that the computation* **c** *does not contain a cycle, and the sequence of pairs is not diverging into still growing pairs. The latter means, that there exist two numbers* $l_1 \in \mathcal{Z}$ *and* $l_2 \in \mathcal{Q}$*, such that for every step* $\langle k, v\rangle$ *of computation* **c***, the inequalities hold* $k < l_1 \wedge v < l_1$.

More details can be found in subsection 7.1.

3. Collatz tree

It is easy to notice that the set of those natural numbers for which the computation of the Collatz algorithm is finite, forms a tree.

Definition 3.1. *Collatz tree* $\mathcal{DC}$ is a subset $D \subset N$ of the set N of natural numbers and the function f defined on the set $D \setminus \{0,1\}$.

$$\mathcal{DC} = \langle D, f\rangle$$

where $D \subset N, 1 \in D, f\colon D \setminus \{0,1\} \to D$.
Function f is determined as follows

$$f(n) = \begin{cases} n \div 2 & \text{when } n \mod 2 = 0 \\ 3n+1 & \text{when } n \mod 2 = 1 \end{cases}$$

[3] A. Tarski [Tar34] confirms that S. Jaśkowski observed (in 1929) that the subset of complex numbers $M \overset{df}{=} \{a+bi \in \mathbb{C}\colon (a \in \mathbb{Z} \wedge b \in \mathbb{Q} \wedge (b \geq 0 \wedge (b = 0 \implies a \geq 0))\}$ satisfies all axioms of Presburger arithmetic.

, the set D is the least set containing the number 1 and closed with respect to the function f,

$$D = \{n \in N : \exists_{i \in N}\ f^i(n) = 1\ \}\ .$$

As it is easy to see, this definition is highly entangled and the decision whether the set D contains every natural number is equivalent to the Collatz problem.

Remark 3.1. Set D has the following properties :

$$x \in D \implies (x + x) \in D \tag{2}$$
$$\big(x \in D \wedge \exists_y x = y + y\big) \implies y \in D \tag{3}$$
$$\big(x \in D \wedge \exists_y x = y + y + 1\big) \implies (x + x + x + 1) \in D \tag{4}$$
$$\big(x \in D \wedge (\exists_e \exists_z e = z + z + 1 \wedge x = e + e + e + 1)\big) \implies e \in D \tag{5}$$

Implications (2) and (5) show left and right son of element x.
Let us note,

Remark 3.2. Let p be an odd , natural number $p = 2j + 1$, then if $p \in D$ then the number $4p + 1$ is an element of the Collatz tree too, $4p + 1 \in D$.

Similar, interesting properties has the complement of set D, **if** it is a non-empty set. Let $cD \stackrel{df}{=} N \setminus D$ denote the complement of set D.

Remark 3.3. If the complement cD is a non-empty set, then it has similar properties:

$$x \in cD \implies (x + x) \in cD \tag{6}$$
$$\big(x \in cD \wedge \exists_y x = y + y\big) \implies y \in cD \tag{7}$$
$$\big(x \in cD \wedge \exists_y x = y + y + 1\big) \implies (x + x + x + 1) \in cD \tag{8}$$
$$\big(x \in cD \wedge (\exists_e \exists_z e = z + z + 1 \wedge x = e + e + e + 1)\big) \implies e \in cD \tag{9}$$

Note, both sets D and cD may be considered as graphs. Their structures are similar. However, the graph cD is not a tree .

Remark 3.4. *If Collatz conjecture is not true, then both sets D and $N \setminus D$ are infinite.*

From properties (6) and (7) follows the

Fact 3.1. If an x element does not belong to the Collatz tree then the computation of the Collatz algorithm starting with the state $v(n) = x$ is not finite.

Let us note the following observation

Remark 3.5. In a non-standard model of elementary theory of natural numbers with addition, the complement of Collatz tree is an infinite, arborescent graph.

Conjecture 3.1. If the Collatz computation of element x is infinite $x \in cD$ then for any natural numbers $n, m > 0$ the computation of $nx \div m$ is infinite too, $(nx \div m) \in cD$.

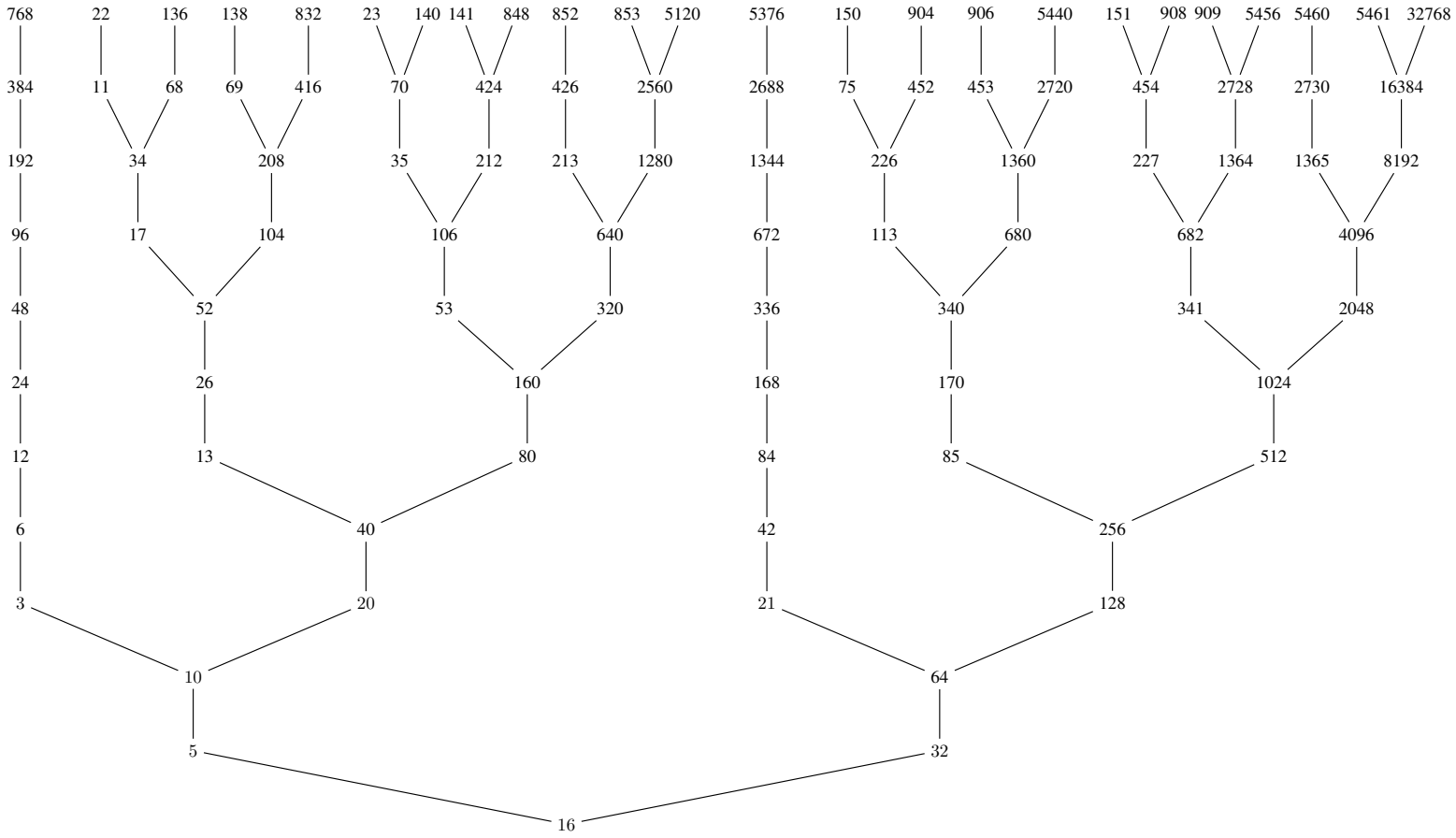


Figure 1. A fragment of Collatz tree, levels 4-15. It does not include levels 0-3, they consist of elements 1 — 2 — 4 — 8 — .

4. Four algorithms, relatives of Cl algorithm

In this section we present an algorithm Gr equivalent to the algorithm Cl and three algorithms $Gr1, Gr2, Gr3$ that are successive extensions of the Gr algorithm.

Lemma 4.1. The following algorithm Gr is equivalent to Collatz algorithm Cl.

Gr:
```
READ(n);
while even(n) do n:= n ÷ 2 od ;
while n ≠1 do
  n:= 3*n+1;
  while even(n) do n:= n ÷ 2 od
od ;
```

Proof:
The equivalence of the algorithms Cl and Gr is intuitive. Compare the recurrence of Collatz (rec1) and the following recurrence (rec2) that is calculated by the algorithm Gr.

$$\left.\begin{array}{rcll} k_0 = exp(n,2) & \wedge & m_0 = \frac{n}{2^{k_0}} & \\ k_{i+1} = exp(3m_i+1,2) & \wedge & m_{i+1} = \frac{3m_i+1}{2^{k_{i+1}}} & \text{for } i \geq 0 \end{array}\right\} \qquad \text{(rec2)}$$

One can say the algorithm Gr is obtained by the elimination of **if** instruction from the Cl algorithm. However, construction of a formal proof is a non-obvious task. We are leaving this task to the reader. □

Next, we present the algorithm $Gr1$, an extension of algorithm Gr.

$Gr1$:
```
var n,l,i :integer ; K, M :arrayof integer;
Γ1: READ(n);  i := 0; l := 0;
    while even(n) do n:= n ÷ 2; l := l + 1 od ; K_i := l; M_i :=n:
while n ≠1 do
Δ1: { M_i=n }  n:= 3*n+1;  l := 0;
    while even(n) do n:= n ÷2; l := l + 1od ; K_{i+1} := l; M_{i+1} :=n;
    {M_{i+1} = (3·M_i+1)/2^{K_i} ∧ K_{i+1} = exp(3 * M_i + 1, 2)}  i := i + 1
od
```

Lemma 4.2. Algorithm $Gr1$ has the following properties:
(*i*) Algorithms Gr and $Gr1$ are equivalent with respect to the halting property.
(*ii*) The sequences $\{M_i\}$ and $\{K_i\}$ calculated by the algorithm $Gr1$ satisfy the recurrence rec2 i.e. for every natural number $i \geq 0$ the equalities $M_i = m_i$ and $K_i = k_i$ hold.

Proof:
Both statements are very intuitive. Algorithm $Gr1$ is an extension of algorithm Gr. The inserted instructions do not interfere with the halting property of algorithm $Gr1$. Second part of the lemma follows easily from the remark that $K_0 = exp(\mathrm{n}, 2)$ and $M_0 = \frac{n}{2^{K_0}}$ and that for all $i > 0$ we have $K_{i+1} = exp(3 * M_i + 1, 2)$ and $M_{i+1} = \frac{3 \cdot M_i + 1}{2^{K_{i+1}}}$. □

Each odd number m in Collatz tree, $m \in D$, initializes a new branch. Let us give a color number $x+1$ to each new branch emanating from a branch with color number x.. Note, for every natural number$p$ the set of branches of the color p is infinite. Let W_x denote the set of natural numbers that obtained the color x.
Besides the levels of Collatz tree, one can distinguish the structure of storeys (or floors) in the tree .

Definition 4.1. Inductive definition of storey (a bunch of branches of the same color).

$$W_0 \stackrel{df}{=} \{n \in N : \exists_{i \in N}\, n = 2^i\}$$

$$W_{x+1} \stackrel{df}{=} \big\{n \in N : \exists_{m \in W_x}\, \exists_{i \in N} \big(n = 3 \cdot \frac{m}{2^{exp(m,2)}} + 1\big) \cdot 2^i\big\}$$

Storey number 0 is composed of all numbers being powers of 2. Different storeys are marked by colors .

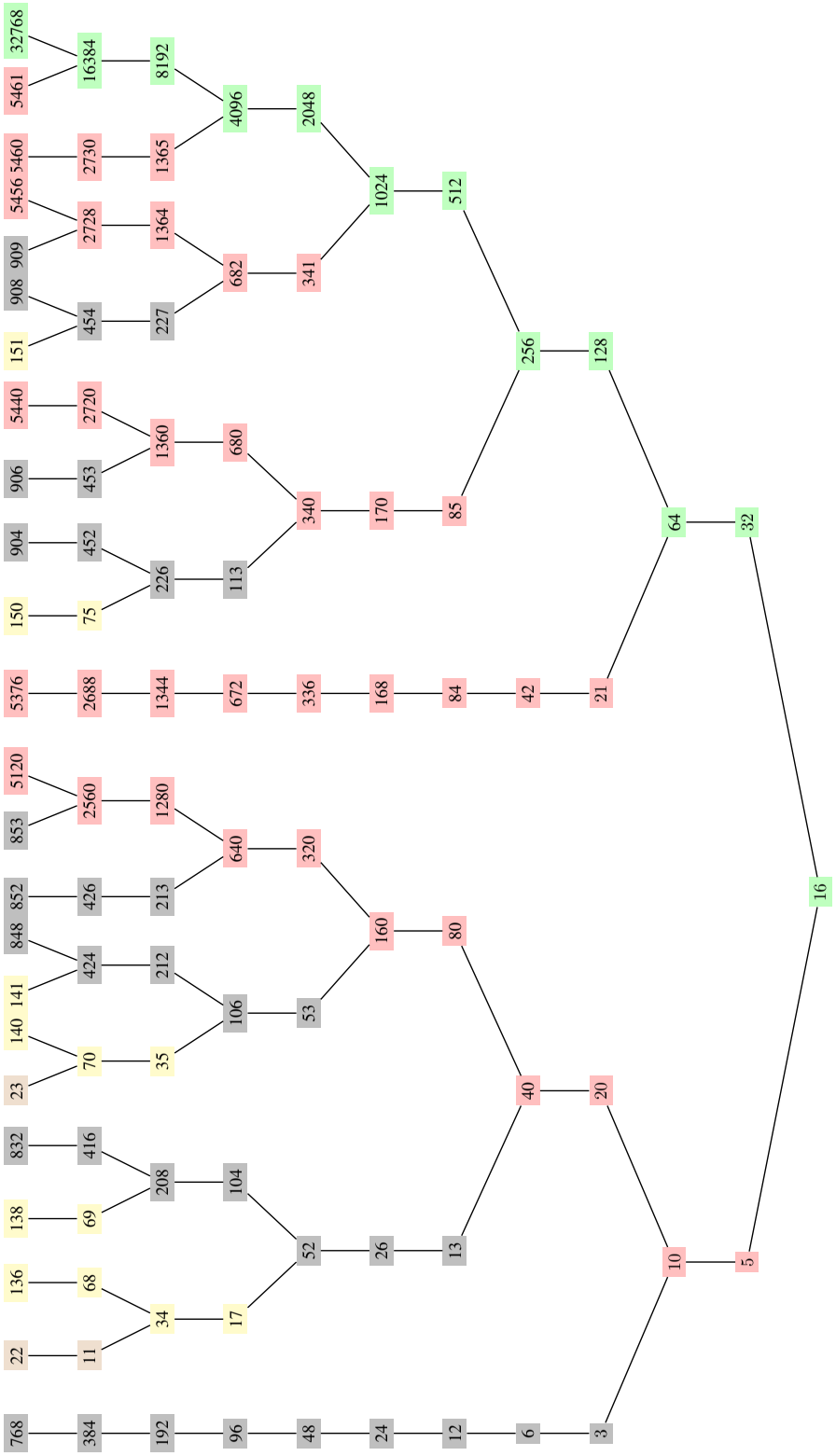


Figure 2. Storeys 0 - 4 of Collatz tree

Let s be a variable not occurring in algorithm $Gr1$. The following lemma states the partial correctnes of the algorithm $Gr1$ w.r.t. precondition $s = n$ and postcondition $s \in W_i$.

Lemma 4.3. Algorithm $Gr1$ computes the number i of storey W_i of number n,

$$\{Gr1\}(true) \implies \big((s = n) \implies \{Gr1\}(s \in W_i)\big)$$

Now, we present another algorithm $Gr2$ and a lemma.

```
var n ,l, i, x, y, z :integer ; k,m :arrayof integer;
Γ2: READ(n);   i := 0; l := 0;
    while even(n) do n:= n ÷ 2; l := l + 1 od ;
    z, k_i := l; m_i:=n; y := 0;
Gr2: while m_i ≠1 do
Δ2:   n:= 3*n+1; i := i + 1; l := 0 ;
      while even(n) do n:= n ÷ 2; l := l + 1 od ;
      k_i := l; m_i:=n; z := z + k_i; y := 3 * y + 2^z; x := i
od
```

Lemma 4.4. Algorithm $Gr2$ has the following properties:
(*i*) Both algorithms $Gr1$ and $Gr2$ are equivalent with respect to the halting property.
(*ii*) Formula φ : $\boxed{n \cdot 3^i + y = m_i \cdot 2^z}$ is an invariant of the program $Gr2$ i.e. the formulas (10) and (11)

$$\{\Gamma_2\}\,(n \cdot 3^i + y = m_i \cdot 2^z) \tag{10}$$

and

$$(n \cdot 3^i + y = m_i \cdot 2^z) \implies \{\Delta_2\}(n \cdot 3^i + y = m_i \cdot 2^z) \tag{11}$$

are theorems of the algorithmic theory of numbers $\mathcal{ATN}$.

Proof:
Proofs of these formulas are easy, it suffices to apply the axiom of assignment instruction Ax_{18}, □

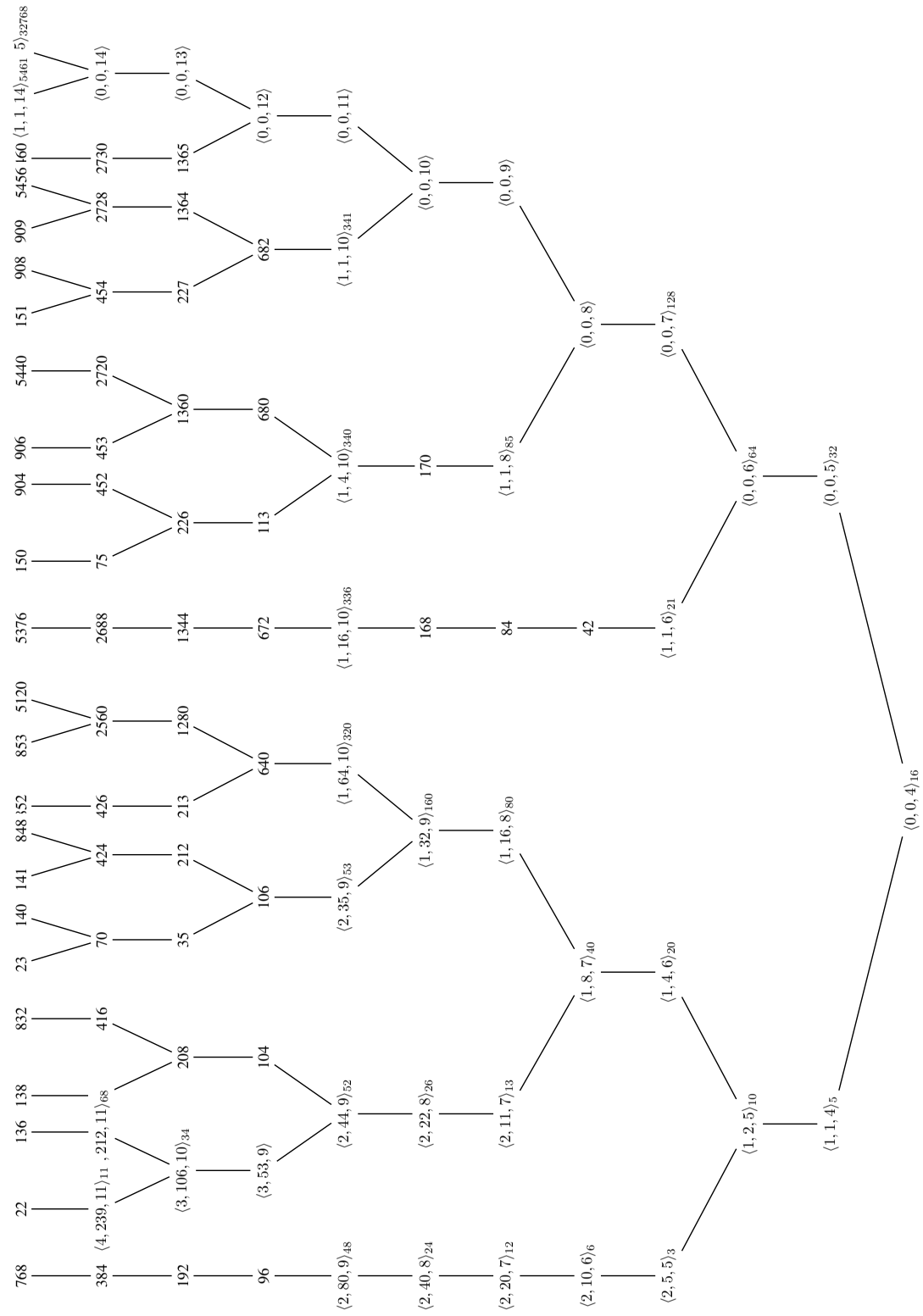


Figure 3. Tree of triples (levels 4 – 15)

Subsequent algorithm $Gr3$ exposes the calculations of x, y, z.

var n,i,aux :integer ; k,m,X,Y,Z : **arrayof** integer;

Γ_3: READ(n); $i := 0$; $k_i := exp(\mathrm{n}, 2)$; $m_i := \frac{\mathrm{n}}{2^{k_i}}$; $Z_i := k_0$; $Y_i, X_i := 0$;

$Gr3$: **while** n $\cdot 3^i + Y_i \neq 2^{Z_i}$ **do**

Δ_3:
aux:=3*m_i+1;
$i := i + 1$;
$k_i := exp(\mathsf{aux}, 2)$; $m_i :=$aux$/2^{k_i}$;
$Y_i := 3Y_{i-1} + 2^{Z_{i-1}}$; $Z_i := Z_{i-1} + k_i$; $X_i := i$;;

od

See some properties of the algorithm $Gr3$.

Lemma 4.5. Both algorithms $Gr2$ and $Gr3$ are equivalent with respect to the halting property.
For every element n after each i-th iteration of algoritm $Gr3$, the following formulas are satisfied

$\varphi: \ n \cdot 3^i + Y_i = m_i \cdot 2^{Z_i}$	$X_i = i$
$Z_i = \sum_{j=0}^{i} k_j$	$Y_i = \sum_{j=0}^{i-1} \left(3^{i-1-j} \cdot 2^{Z_j} \right)$

where the sequences $\{m_i\}$and $\{k_i\}$ are determined by the recurrence (rec2).
In other words

$$\Gamma_3 \bigcap \{\mathbf{if}\ m_i \neq 1\ \mathbf{then}\ \Delta_3\ \mathbf{fi}\}\, \varphi$$

Remark 4.1. Hence, for every element n algorithm $Gr3$ calculates an increasing, monotone sequence of triples $\langle i\,(= X_i), Y_i, Z_i \rangle$.

We can say informally that the algorithm $Gr3$ performs as follow

$i := 0$;
while $n \notin W_i$ **do** $i := i + 1$ **od**

Note, one test $n \cdot 3^i + Y_i \neq 2^{Z_i}$ suffices to assert that $n \notin W_i$. There is at most (only) one way from n to W_0.

Hotel Collatz

The next figure 4 seems to be somewhat chaotic. Many have heard of Hilbert Hotel. This is an implementation of the Hilbert's idea. Imagine, we have to disposition a quarter of infinite plane of land. Our architect envisioned an inifinite set of corridors, or are they towers. Hotel contains rooms of any natural number. Let $n = 2^i \cdot (2j + 1)$. It means that the room number n is located in tower number j on the floor number i . Each tower is equipped with an elevator (shown as a green line). Moreover, each tower is connected to another by a staircaise that connects numbers $k = 2j + 1$ and $3k + 1$. This is shown as a red arrow $\overrightarrow{\langle k, 3k + 1\rangle}$. Note that the red line is drawn if

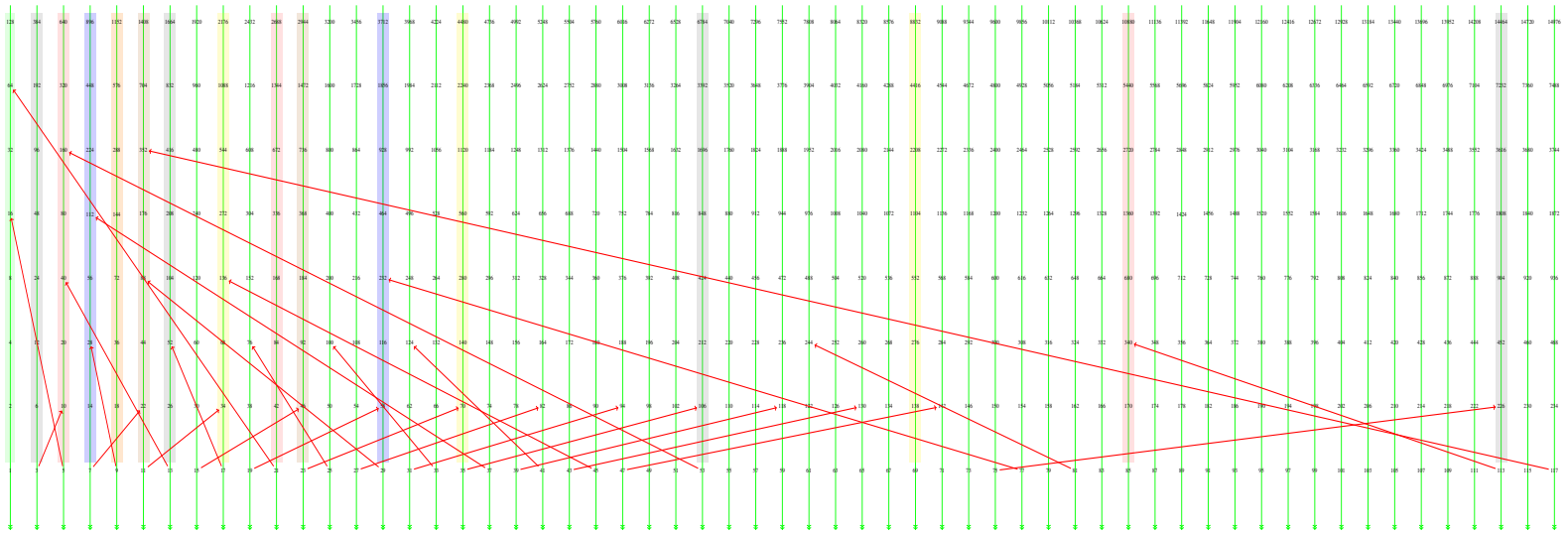


Figure 4. Hotel Collatz

and only if both of its ends lie inside the drawn fragment of the Collatz hotel. Thus, many lines are omitted. The full (infinite) picture contains infinitely many of red arrows.

Definition 4.2. The graph $HC = \langle V, E\rangle$ is defined as follows

$V = N$ i.e. the set of vertices is the set of standard, reachable, natural numbers

$E = \{\overrightarrow{\langle k, p\rangle} : \exists_p\, k = p + p\} \cup \{\overrightarrow{\langle k, 3k + 1\rangle} : \exists_p k = p + p + 1\}$ edges of the graph

Conjecture 4.1. The hotel Collatz is an infinite, connected, acyclic graph, i.e it is a tree. Number 1 is the root.

5. On finite and infinite computations of Collatz algorithm

QUESTION Can some computation contain both reachable and unreachable elements?
No. The subset of reachable elements is closed with respect of division by 2 and multiplication by 3. The same observation applies to the set of unreachable elements.
We know, cf. subsection 7.1 that computations of nonreachable elements are infinite. Therefore, it suffices to show that any infinite computation requires that a non-standard element is present in it.

5.1. Finite computations

Let $\mathfrak{M} = \langle M; 0, 1, +, = \rangle$ be any algebraic structure that is a model of elementary theory of addition of natural numbers, c.f. subsection 7.2.
Denotation. Let $\theta(x, y)$ be a formula. The expression $(\mu x)\theta(x, y)$ denote the least element $x(y)$ such that the value of the formula is truth.
EXAMPLE. $(\mu x)(x + x > y)$.

Lemma 5.1. Let n be an arbitrary element of the structure $\mathfrak{M}$. The following conditions are equivalent

(*i*) The sequence $n_0 = n$ and $n_{i+1} = \begin{cases} n_i \div 2 & \text{when } n_i \mod 2 = 0 \\ 3n_i + 1 & \text{when } n_i \mod 2 = 1 \end{cases}$ determined by the recurrence (rec1) contains an element $n_j = 1$

(*ii*) The computation of the algorithm Cl is finite.

(*iii*) The sequence $m_0 = \frac{n}{2^{k_0}}$ and $m_{i+1} = \frac{3m_i+1}{2^{k_i}}$ determined by the recurrence (rec2) stabilizes, i.e. there exist l such that $m_k = 1$ for all $k > l$

(*iv*) The computation of the algorithm Gr is finite.

(*v*) The computation of the algorithm $Gr1$ is finite and the subsequent values of the variables M_i and K_i satisfy the recurrence (rec2) .

(*vi*) The computation of the algorithm $Gr2$ is finite and the subsequent values of the variables m_i and k_i satisfy the recurrence (rec2). The formula $n \cdot 3^x + y = m_i \cdot 2^z$ holds after each iteration of **while** instruction, i.e. it is the invariant of the program Δ_2. The final valuation of variables x, y, z and n satisfies the equation $n \cdot 3^x + y = 2^z$.

(*vii*) The computation of the algorithm $Gr3$ is finite.
The subsequent values of the variables m_i and k_i satisfy the recurrence (rec2) .
The subsequent values of the variables X_i, Y_i, Z_i form a monotone, increasing sequence of triples.
The formula $n \cdot 3^{X_i} + Y_i = m_i \cdot 2^{Z_i}$is satisfied at each i-th iteration of the program $Gr3$.

The proof of this lemma is left to the reader.
Suppose that for a given element n the computationof algorithm $Gr2$ is finite.
Let $\bar{x} = (\mu x)\left(n \cdot 3^x + \left[\sum_{j=0}^{x-1}\left(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)\right] = 2^{\sum_{j=0}^{x} k_j}\right)$. Put $\bar{y} = \sum_{j=0}^{\bar{x}-1}\left(3^{\bar{x}-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)$ and

$\bar{z} = \sum_{j=0}^{\bar{x}} k_j$.

We present the algorithm IC', which is a slightly modified version of the algorithm IC devised in [MS21] .

$$IC' : \left\{ \begin{array}{l} \textbf{var}\ \text{x,y,z},\ j : integer, Err : Boolean; \\ \text{READ(x,y,z)};\quad j := \text{x};\ Err := false; \\ \textbf{while}\ \ \text{x+y+z} \neq 0\ \ \textbf{do} \\ Tr : \boxed{\begin{array}{l} \textbf{if}\ \big(odd(\text{y}) \wedge ((\text{x}= 0)\ \text{or}\ (\text{y}< 3^{\text{x}-1}))\big)\ \textbf{then}\ Err := true;\ \textbf{exit fi}; \\ \textbf{if}\ even(\text{y})\ \textbf{then}\ \text{y}:= \frac{\text{y}}{2^{k_{j-\text{x}}}}; \text{z}:=\text{z}-k_{j-\text{x}}\ \textbf{else}\ \text{x}:=\text{x}-1;\text{y}:=\text{y}-3^{\text{x}}\ \textbf{fi} \end{array}} \\ \textbf{od} \end{array} \right\}$$

The algorithm makes use of the elements k_i that are determined by the recurrence (rec2). We observe the following fact

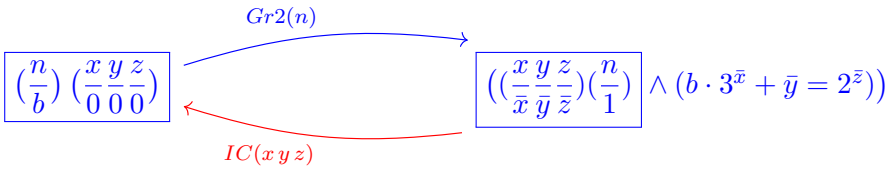
Lemma 5.2. For every element n

$$(n = b) \wedge \{Gr2\}\Big(\big((x = \bar{x} \wedge y = \bar{y} \wedge z = \bar{z}) \wedge (b \cdot 3^{\bar{x}} + \bar{y} = 2^{\bar{z}})\big) \Rightarrow \{IC'\}(x = y = z = 0)\Big)$$

and

$$(x = \bar{x} \wedge y = \bar{y} \wedge z = \bar{z}) \wedge (b \cdot 3^{\bar{x}} + \bar{y} = 2^{\bar{z}})\big) \wedge \Big(\{IC'\}(x = y = z = 0) \implies (n = b) \implies \{Gr2\}(x = \bar{x} \wedge y = \bar{y}\ land z = \bar{z})\Big)$$

The contents of this lemma is best explained by the commutativity of the diagram below.



Proof:

The proof makes use of two facts:

1) $even(n) \equiv even(\sum_{j=0}^{x-1} \big(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\big))$

2) $\sum_{j=0}^{x-1} \big(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\big) = 2^{k_0} \cdot (3^{x-1} + 2^{k_1} \cdot (3^{x-2} + 2^{x-3} \cdot (\cdots + 2^{k_x} \cdot 3^0)))$

One can prove this lemma by induction w.r.t. number of encountered odd numbers.

The thesis of the lemma is very intuitive. Look at the Collatz hotel Fig. 4. The lemma states that for every room number n the two conditions are equivalent (1) there is a path from room number n to the

exit, which is located near room number 1, (2) there is a path from entry to the hotel near room number 1 to the room number n. It is clear that such a path must be complete, no jumps are allowed.

□

Lemma 5.3. For every element n the following conditions are equivalent

(*i*) computation of Collatz algorithm Cl is finite,

(*ii*) there exists the **least** element x such that the following equality holds

$$n \cdot 3^x + \left(\sum_{j=0}^{x-1} \left(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l} \right) \right) = 2^{\sum_{j=0}^{x} k_j}. \tag{Mx}$$

Note, If there is the least element x satisfying the equation (Mx) then it is a reachable natural number. Denote this element x_0. The length $L(n)$ of the computation is then $x_0 + \sum_{j=0}^{x_0} k_j$. Which means: x_0 is the number of multiplication by 3, $z = \sum_{j=0}^{x} k_j$ is the total number of divisions by 2 and for every $0 \leq j \leq x-1$ the number k_j is the number of divisions by 2 excuted in between the j-th and $j+1$-th execution of multiplication by 3.

The algorithm Cl executes $x + z$ iterations.

Proof:

We shall illustrate and summarize the considerations on finite computations in the following commutative diagram. □

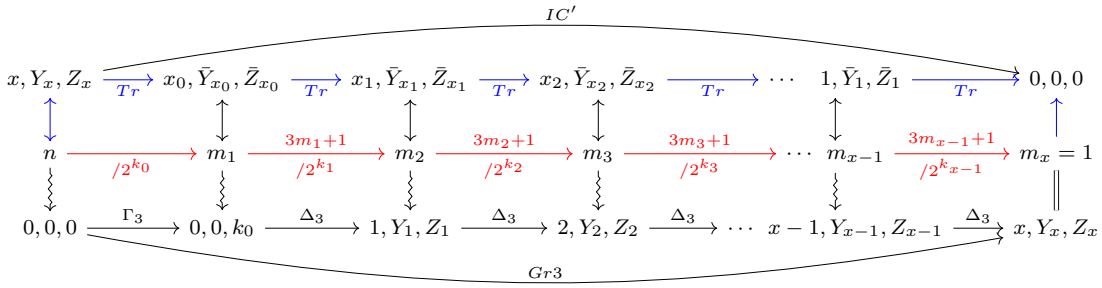


Figure 5. CASE OF FINITE COMPUTATION EXEMPLIFIED

Middle row, (with red arrows) represents computation of $Gr1$, elements k_i and m_i are given by the recurrence (rec2)
third row shows computation of $Gr3$, the subsequent triples are $X_{i+1} = i+1, Y_{i+1} = 3Y_i + 2^{Z_i}_i, Z_{i+1} = Z_i + k_i$
first row (blue arrows) shows computation of algorithm IC on triples, $\bar{Y}_x = Y_x$ and $\bar{Z}_x = Z_x$ and for $i = x, \ldots, 1$ we have $\bar{Z}_{i-1} = \bar{Z}_i - k_i$ and $\bar{Y}_{i-1} = (\bar{Y}_i / 2^{k_i}) - 3^{i-1}$

What happens if someone executes one more iteration of algorithm Δ_3?

$$m_{x+1} = 1 \wedge k_{x+1} = 2 \wedge X_{x+1} = x+1 \wedge Y_{x+1} = 3Y_x + 2^{Z_x} \wedge Z_{x+1} = Z_x + k_{x+1}$$

and we see that the equality $n \cdot 3^{x+1} + Y_{x+1} = 2^{Z_{x+1}}$ holds again. This explains and justifies the use of the word "*least*".

5.2. Infinite computations

Lemma 5.4. Let n be an arbitrary element of the structure $\mathfrak{M}$. The following conditions are equivalent

(*i*) The computation of the algorithm Cl is infinite.

(*ii*) The computation of the algorithm Gr is infinite.

(*iii*) The computation of the algorithm $Gr1$ is infinite and the subsequent values of the variables m_i and k_i satisfy the recurrence (rec2) . and for every $i > 0$ the element m_i is not equal 1.

(*iv*) The computation of the algorithm $Gr2$ is infinite and the subsequent values of the variables m_i and k_i satisfy the recurrence (rec2) . and for every $i > 0$ the element m_i is not equal 1.
Moreover, for every $i > 0$ the formula $n \cdot 3^i + y = m_i \cdot 2^z$ is invariant of algorithm $Gr2$.

(*v*) The computation of the algorithm $Gr3$ is infinite and the subsequent values of the variables m_i and k_i satisfy the recurrence (rec2) . and for every $i > 0$ the element m_i is not equal 1.
After every i-th iteration of the algorithm the equation $n \cdot 3^{X_i} + Y_i = m_i \cdot 2^{Z_i}$ is satisfied by the current valuation of variables X_i, Y_i, Z_i. The triples $\langle X_i, Y_i, Z_i\rangle$ form an infinite, monotone, increasing sequence of *reachable* numbers. Therefore, there is no a triple $\langle X_i, Y_i, Z_i\rangle$ such that the equation $n \cdot 3^{X_i} + Y_i = 2^{Z_i}$ is satisfied.

Lemma 5.5. If for a certain element n the computation is infinite then n is not a reachable number.

Proof:
If the computation for an element n is infinite then it is uniquely determined by its origin, i.e. the number n. This is easily seen from the diagram below.

$$\begin{array}{ccccccccc}
n & \xrightarrow[/2^{k_0}]{} & m_1 & \xrightarrow[/2^{k_1}]{3m_1+1} & m_2 & \cdots & \to m_{x-1} & \xrightarrow[/2^{k_x}]{3m_{x-1}+1} & m_x \neq 1 \to \quad \underline{\cdots \to} \\
\| & & \| & & \| & & \| & & \| \\
\langle 0,0,0\rangle & \xrightarrow[\Gamma_3]{} & \langle 0,0,k_0\rangle & \xrightarrow[\Delta_3]{} & \langle 1,1,,k_0\rangle & \cdots & \to \langle x-1, Y_{x-1}, Z_{x-1}\rangle & \xrightarrow[\Delta_3]{} & \langle x, Y_x, Z_x\rangle \to \quad \underline{\cdots \to}
\end{array}$$

The sequence of triples

$$\left\langle i,\ \sum_{j=0}^{i-1}\left(3^{i-1-j}\cdot 2^{\sum_{l=0}^{j} k_l}\right),\ \sum_{j=0}^{i} k_j \right\rangle_{i\in Nat}$$

is unique, infinite computation for n. By lemma 5.3 we see that the value of the expression

$$(\mu x)\left(n\cdot 3^x + \left(\sum_{j=0}^{x-1}\left(3^{x-1-j}\cdot 2^{\sum_{l=0}^{j} k_l}\right)\right) = 2^{\sum_{j=0}^{x} k_j}\right)$$

is undefined. This means that,
there is no triple $\langle \bar{x}, \bar{y}, \bar{z}\rangle$ such that

- the triple $\langle \bar{x}, \bar{y}, \bar{z}\rangle$ is the *least* triple satisfying the equation $n \cdot 3^{\bar{x}} + \bar{y} = 2^{\bar{z}}$

and simultaneously

- the algorithm IC' starting with the numbers $\bar{x}, \bar{y}, \bar{z}$ terminates.

On the other hand, from the lemma 7.4, we see that there is a triple $\langle x_n, y_n, z_n\rangle$ of elements that satisfy the equation $n \cdot 3^x + y = 2^z$ and for every reachable element $\mathbf{i} \in Nat$ the triple
$\langle \mathbf{i}, \sum\limits_{j=0}^{\mathbf{i}-1}\left(3^{\mathbf{i}-1-j} \cdot 2^{\sum\limits_{l=0}^{j} k_l}\right), \sum\limits_{j=0}^{\mathbf{i}} k_j\rangle$ provided by the algorithm $Gr3$ differs from the triple $\langle x_n, y_n, z_n\rangle$.
As a consequence, the computation of algorithm IC' starting from triple $\langle x_n, y_n, z_n\rangle$ is a decreasing infinite sequence of triples.

$$\langle x_n, y_n, z_n\rangle \xrightarrow[Tr]{} \langle x_n - 1, \bar{Y}_{x_n-1}, \bar{Z}_{x_n-1}\rangle \xrightarrow[Tr]{} \langle x_n - 2, \bar{Y}_{x_n-2}, \bar{Z}_{x_n-2}\rangle \xrightarrow[Tr]{} \cdots$$

(It can not be finite in the view of lemma 5.3 .)
This means that elements x_n, z_n, y_n all are unreachable. For $x_n \notin Nat$ and $y_n > x_n$ and $z_n > x_n$. Hence by the remark 7.1 the whole computation is entirely contained in the set of unreachable elements. By the lemma 7.6 we obtain that element n is unreachable. □

5.3. Collatz theorem

Consider the structure $\mathfrak{N}$ of reachable natural numbers.

Theorem 5.1. Let n be any standard (reachable) element od the structure $\mathfrak{N}$. The computation of Collatz algorithm Cl that begins with n is finite.

Proof:
The proof follows immediately from the lemmas 5.3 and 5.5. □

Corollary 5.1. Conjectures 3.1 and 4.1 formulated above are valid statements.

6. Final remarks

Reportedly, Pál Erdős said: "mathematics is not ripe enough to solve this problem (Collatz)". We disagree. In our opinion a consortium of Alfred Tarski, Kurt Goedel and Stephen Kleene was able to solve the Collatz conjecture in 1937.
Andrzej Mostowski expected that many arithmetic theorems independent of the Peano axioms should be found. Here is one example. The theorem on termination of Euclid's algorithm is another example of a theorem which is valid and unprovable in Peano theory.. Note, both theorems need to be stated as algorithmic formulas, there is no first-order formula that expresess the termination property of Euclid's algorithm or Collatz algorithm.

We hope the reader will forgive us for a moment of insistence (is it an agitpropaganda?) .
Calculus of programs $\mathcal{AL}$ is a handy tool. For there are some good reasons to use the calculus of programs

(*i*) The language of calculus $\mathcal{AL}$ contains algorithms (programs) and *algorithmic formulas* besides terms and elementary.

(*ii*) Any semantical property of an algorithm can be *expressed* by an appropriate algorithmic formula. Be it termination, correctness or other properties.

(*iii*) Algorithmic formulas enable to create complete, categorical *specifications* of data structures in the form of algorithmic theories.

(*iv*) Calculus of programs $\mathcal{AL}$ brings a *complete* set of tools for proving theorems of algorithmic theories.

The contribution presented here leaves some open questions: first of all the cost of the algorithm Cl remains to be estimated. The lower bound is obviously $O(x + z)$. A tight upper bound remains to be found .

Another goal, that will take more time, is to write a complete syntactical (i.e. free of any semantical considerations, like studies of computation) proof of Collatz theorem. We expect that the proof will pass the checking by a proof-checker proper for calculus of programs $\mathcal{AL}$ [4]. The subsections 7.5 and 7.6 contribute to this work.

Acknowlegments

7. Suplements

For the reader's convenience, in this section we have included some definitions, some useful theorems, and samples of proofs in algorithmic natural number theory.

7.1. A structure with counterexamples

where Collatz computations may be of infinite length

Here we present some facts that are less known to the IT community.
These facts may seem strange. The reader may doubt the importance of those facts. Yet, it is worth considering, strange data structures do exist, and this fact has ramifications. Strange as they seem, still it is worthwhile to be aware of their existence.
Now, we will expose the algebraic structure $\mathfrak{M}$, which is a model of the theory Ar, i.e. all axioms of theory Ar are true in the structure $\mathfrak{M}$. First we will describe this structure as mathematicians do, then we will write a class (ie a program module) implementing this structure. medskip

Mathematical description of the structure

$\mathfrak{M}$ is an algebraic structure

$$\mathfrak{M} = \langle M;\ \underline{\mathbf{0}}, \underline{\mathbf{1}}, \oplus;\ = \rangle \qquad \text{(NonStandard)}$$

[4] such proof-checker does not exists yet and it is to be built

such that M is a set of complex numbers $k + \imath w$, i.e. of pairs $\langle k, w\rangle$, where element $k \in \mathbb{Z}$ is an integer, and element $w \in \mathbb{Q}^+$ is a rational, non-negative number $w \geq 0$ and the following requirements are satisfied:

(*i*) for each element $k + \imath w$ if $w = 0$ then $k \geq 0$,

(*ii*) $\underline{\mathbf{0}} \stackrel{df}{=} \langle 0 + \imath 0\rangle$,

(*iii*) $\underline{\mathbf{1}} \stackrel{df}{=} \langle 1. + \imath 0\rangle$,

(*iv*) the operation $\oplus$ of addition is determined as usual

$$(k + \imath w) \oplus (k' + \imath w') \stackrel{df}{=} (k + k') + \imath(w + w').$$

(*v*) the predicate = denotes as usual identity relation.

Lemma 7.1. The algebraic structure $\mathfrak{M}$ is a model of first-order arithmetic of addition of natural numbers $\mathcal{T}$, cf. next subsection 7.2

The reader may check that every axiom of the $\mathcal{T}$ theory (see definition7.2, p.24), is a sentence true in the structure $\mathfrak{M}$.

The substructure $\mathfrak{N} \subset \mathfrak{M}$ composed of only those elements for which $w = 0$ is also a model of the theory $\mathcal{T}$.

It is easy to remark that elements of the form $\langle k, 0\rangle$ may be identified with natural numbers k, $k \in N$. Have a look at table 1

The elements of the structure $\mathfrak{N}$ are called *reachable*, for they enjoy the following algorithmic property

$$\forall_{n\in N}\, \{y := \mathbf{0}; \mathbf{while}\ y \neq n\ \mathbf{do}\ y := y + \mathbf{1}\ \mathbf{od}\}(y = n)$$

The structure $\mathfrak{M}$ is not a model of the $\mathcal{ATN}$, algorithmic theory of natural numbers, cf . subsection 7.4. .Elements of the structure $\langle k, w\rangle$. such as $w \neq \mathbf{0}$ are *unreachable*. i.e. for each element $x_0 = \langle k, w\rangle$ such that $w \neq 0$ the following condition holds

$$\neg\{y := \mathbf{0}; \mathbf{while}\ y \neq x_0\ \mathbf{do}\ y := y + \mathbf{1}\ \mathbf{od}\}(y = x_0)$$

The subset $\mathfrak{N} \subset \mathfrak{M}$ composed of only those elements for which $w = 0$ is a model of the theory $\mathcal{ATN}$ c.f. subsection 7.4. The elements of the structure $\mathfrak{N}$ are called *reachable*. A very important theorem of the foundations of mathematics is

Fact 7.1. The structures $\mathfrak{N}$ and $\mathfrak{M}$ are not isomorphic. See [Grz71], p. 256.

As we will see in a moment, this fact is also important for IT specialists.

An attempt to visualize structure $\mathfrak{M}$ is presented in the form of table 1.

Table 1. Fragment of structure $\mathfrak{M}$

STANDARD (*reachable*) elements	*Unreachable* (INFINITE) elements						
				$\cdots$			
	$-\infty\cdots$	$-11+\imath 2$	$-10+\imath 2$	$\cdots$	$0+\imath 2$	$1+\imath 2$	$2+\imath 2$	$\cdots\infty$
				$\cdots$			
	$-\infty\cdots$	$-11+\imath\frac{53}{47}$	$-10+\imath\frac{53}{47}$	$\cdots$	$0+\imath\frac{53}{47}$	$1+\imath\frac{53}{47}$	$2+\imath\frac{53}{47}$	$\cdots\infty$
				$\cdots$			
	$-\infty\cdots$	$-11+\imath\frac{28}{49}$	$-10+\imath\frac{28}{49}$	$\cdots$	$0+\imath\frac{28}{49}$	$1+\imath\frac{28}{49}$	$2+\imath\frac{28}{49}$	$\cdots\infty$
				$\cdots$			
	$-\infty\cdots$	$-11+\imath\frac{3}{47}$	$-10+\imath\frac{3}{47}$	$\cdots$	$0+\imath\frac{3}{47}$	$1+\imath\frac{3}{47}$	$2+\imath\frac{3}{47}$	$\cdots\infty$
				$\cdots$			
$0\ \ 1\ \ 2\ \ \cdots\ \ 101\ \ \cdots\ \ \infty$							

The universe of the structure $\mathfrak{M}$ decomposes onto two disjoint subsets (one green and one red). Every element of the form $\langle k, 0\rangle$ (in this case $k > 0$) represents the natural number k. Such elements are called *reachable* ones. Note,

Definition 7.1. An element n is a standard natural number (i.e. is *reachable*) iff the program of adding ones to initial zero terminates

$$n \in N \overset{df}{\Leftrightarrow} \{q := \mathbf{0};\ \mathbf{while}\ q \neq n\ \mathbf{do}\ q := q + \mathbf{1}\ \mathbf{od}\}(q = n)$$

or, equivalently

$$n \in N \overset{df}{\Leftrightarrow} \{q := \mathbf{0}\} \bigcup \{\mathbf{if}\ n \neq q\ \mathbf{then}\ q := q + \mathbf{1}\ \mathbf{fi}\}(q = n)$$

The other elements are called *unreachable*. Note that the subset that consists of all non-reachable elements is well separated from the subset of reachable elements. Namely, every reachable natural number is less that any unreachable one. Moreover, there is no least element in the set of unreachable elements. I.e. the principle of minimum does not hold in the structure $\mathfrak{M}$.
Moreover, for every element n its computation contains either only standard, reachable numbers or is composed of only unreachable elements. This remark will be of use in our proof.

Remark 7.1. For every element n the whole computation is either in green or in reed quadrant.

Elements of the structure $\mathfrak{M}$ are ordered as usual

$$\forall_{x,y}\ x < y \overset{df}{=} \exists_{z \neq \mathbf{0}}\ x + z = y.$$

Therefore, each reachable element is smaller than every unreachable element.
The order defined in this way is the lexical order. (Given two elements p and q, the element lying higher is bigger, if both are of the same height then the element lying on the right is bigger.)
The order type is $\omega + (\omega^* + \omega) \cdot \eta$

Remark 7.2. The subset of unreachable elements (red) does not obey the principle of minimum.

Definition in programming language

Perhaps you have already noticed that the $\mathfrak{M}$ is a computable structure. The following is a class that implements the structure $\mathfrak{M}$. The implementation uses the integer type, we do not introduce rational numbers explicitly.

```
unit StrukturaM: class;
  unit Elm: class(k,li,mia: integer);
  begin
    if mia=0 then raise Error fi;
    if li * mia <0 then raise Error fi;
    if li=0 and k<0 then raise Error fi;
  end Elm;
  add: function(x,y:Elm): Elm;
  begin
    result := new Elm(x.k+y.k, x.li*y.mia+x.mia*y.li, x.mia*y.mia )
  end add;
  unit one : function:Elm;  begin   result:= new Elm(1,0,2)  end one;
  unit zero : function:Elm;  begin   result:= new Elm(0,0,2)  end zero;
  unit eq: function(x,y:Elm): Boolean;
  begin
    result := (x.k=y.k) and (x.li*y.mia=x.mia*y.li )
  end eq;
end StrukturaM
```

The following lemma expresses the correctness of the implementation with respect to the axioms of Presburger arithmetic (c.f. subsection 7.2) treated as a specification of a class (module of program).

Lemma 7.2. The structure $\mathfrak{E} = \langle E, add, zero, one, eq\rangle$ composed of the set $E = \{o \text{ object} : o\ inElm\}$ of objects of class Elm with the *add* operation is a model of the Ar theory,

$$\mathfrak{E} \models Ar$$

Infinite Collatz algorithm computation

How to execute the Collatz algorithm in StructuraM? It's easy.

```
pref StrukturaM block
  var n: Elm;
  unit odd: function(x:Elm): Boolean; ... result:=(x.k mod 2)=1 ... end odd;
  unit div2: function(x:Elm): Elm; ...
  unit 3xp1: function(n: Elm): Elm; . . . result:=add(n,add(n,add(n,one))); . . . end 3xp1;
begin
  n:= new Elm(8,1,2);
  CI:  while  not eq(n,one)  do
         if  odd(n) then
           n:=3xp1(n)  else n:= div2(n)     (* a version of algorithm CI that uses class Elm *)
       fi
       od
end block;
```

Below we present the computation of Collatz algorithm for $n = \langle 8, \frac{1}{2}\rangle$.

$$\langle 8, \frac{1}{2}\rangle,\ \langle 4, \frac{1}{4}\rangle, \langle 2, \frac{1}{8}\rangle,\ \langle 1, \frac{1}{16}\rangle,\ \langle 4, \frac{3}{16}\rangle,\ \langle 2, \frac{3}{32}\rangle,\ \langle 1, \frac{3}{64}\rangle,\ \langle 4, \frac{9}{64}\rangle,\ \langle 2, \frac{9}{128}\rangle, \cdots$$

Note, the computation of algorithm Gr for the same argument, looks simpler

$$\langle 8, \frac{1}{2}\rangle,\ \langle 4, \frac{1}{4}\rangle, \langle 2, \frac{1}{8}\rangle,\ \langle 1, \frac{1}{16}\rangle,\ \langle 1, \frac{3}{64}\rangle,\ \langle 1, \frac{9}{256}\rangle, \cdots$$

None of the elements of the above sequence is a standard natural number. Each of them is unreachable. It is worth looking at an example of another calculation. Will something change when we assign n a different object? e.g. n: = new Elm (19,2,10)?

$\langle 19, \frac{10}{2}\rangle,\ \langle 58, \frac{30}{2}\rangle, \langle 29, \frac{30}{4}\rangle,\ \langle 88, \frac{90}{4}\rangle,\ \langle 44, \frac{90}{8}\rangle,\ \langle 22, \frac{90}{16}\rangle,\ \langle 11, \frac{90}{32}\rangle,\ \langle 34, \frac{270}{32}\rangle,\ \langle 17, \frac{270}{64}\rangle,$
$\langle 52, \frac{810}{64}\rangle, \langle 26, \frac{405}{64}\rangle, \langle 13, \frac{405}{128}\rangle, \langle 40, \frac{1215}{128}\rangle, \langle 20, \frac{1215}{256}\rangle, \langle 10, \frac{1215}{256}\rangle, \langle 5, \frac{1215}{512}\rangle, \langle 16, \frac{3645}{512}\rangle, \langle 8, \frac{3645}{1024}\rangle,$
$\langle 4, \frac{3645}{2048}\rangle, \langle 2, \frac{3645}{4096}\rangle, \langle 1, \frac{3645}{8192}\rangle, \langle 4, \frac{3*3645}{8192}\rangle, \langle 2, \frac{3645*3}{2*8192}\rangle, \langle 1, \frac{3*3645}{4*8192}\rangle, \langle 4, \frac{9*3645}{4*8192}\rangle, \cdots$

And one more computation.

$\langle 19, 0\rangle,\ \langle 58, 0\rangle, \langle 29, 0\rangle,\ \langle 88, 0\rangle,\ \langle 44, 0\rangle,\ \langle 22, 0\rangle,\ \langle 11, 0\rangle,\ \langle 34, 0\rangle,\ \langle 17, 0\rangle, \langle 52, 0\rangle, \langle 26, 0\rangle,$
$\langle 13, 0\rangle, \langle 40, 0\rangle, \langle 20, 0\rangle, \langle 10, 0\rangle, \langle 5, 0\rangle, \langle 16, 0\rangle, \langle 8, 0\rangle,\ \langle 4, 0\rangle, \langle 2, 0\rangle, \langle 1, 0\rangle.$

Corollary 7.1. The structure $\mathfrak{M}$, which we have described in two different ways, is the model of the Ar theory (you can also say that this structure implements the specification given by the axioms of the Ar theory), with the non-obvious presence of unreachable elements in it.

Another observation

Corollary 7.2. The halting property of the Collatz algorithm cannot be proved from the axioms of the $\mathcal{T}$ theory, nor from the axioms of Ar theory.

7.2. Presburger arithmetic

Presburger arithmetic is another name of elementary theory of natural numbers with addition.
We shall consider the following theory , cf. [Pre29],[Grz71] p. 239 and following ones.

Definition 7.2. Theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, Ax\rangle$ is the system of three elements:

- $\mathcal{L}$ is a language of first-order. The alphabet of this language consist of: the set V of variables, symbols of operations: $0, S, +$, symbol of equality relation $=$, symbols of logical functors and quantifiers, auxiliary symbols as brackets ...
  The set of well formed expressions is the union of te set T of terms and the set of formulas F.
  The set T is the least set of expressions that contains the set V and constants 0 and 1 and closed with respect to the rules: if two expressions τ_1 and τ_2 are terms, then the expression $(\tau_1 + \tau_2)$ is a term too.
  The set F of formulas is the least set of expressions that contains the equalities (i.e. the expressions

of the form $(\tau_1 = \tau_2)$) and closed with respect to the following formation rules: if expressions α and β are formulas, then the aexpression of the form

$$(\alpha \vee \beta),\ (\alpha \wedge \beta),\ (\alpha \implies \beta),\ \neg\alpha$$

are also formulas, moreover, the expressions of the form

$$\forall_x\, \alpha,\ \exists_x\, \alpha$$

where x is a variable and α is a formula, are formulas too.

$\mathcal{C}$ is the operation of consquence determined by axioms of first-order logic and the inference rules of the logic,

Ax is the set of formulas listed below.

$\forall_x\ x + 1 \neq 0$ (a)

$\forall_x \forall_y\ x + 1 = y + 1 \implies x = y$ (b)

$\forall_x\ x + 0 = x$ (c)

$\forall_{x,y}\ (y + 1) + x = (y + x) + 1$ (d)

$\Phi(0) \wedge \forall_x\, [\Phi(x) \implies \Phi(x + 1)] \implies \forall_x \Phi(x)$ (I)

The expression $\Phi(x)$ may be replaced by any formula. The result is an axiom of theory This is the induction scheme.
We augment the set of axioms adding four axioms that define a coiple of useful notions.

$even(x) \stackrel{df}{\equiv} \exists_y\, x = y + y$ (e)

$odd(x) \stackrel{df}{\equiv} \exists_y\, x = y + y + 1$ (o)

$x\, div\, 2 = y \equiv (x = y + y \ \vee\ x = y + y + 1)$ (D2)

$3x \stackrel{df}{=} x + x + x$ (3x)

The theory obtained in this way is a conservative extension of theory $\mathcal{T}$.

Below we present another theory $\mathcal{A}r$ c.f. [Pre29], we shall use two facts: 1) theory $\mathcal{A}\nabla$ is complete and hence is decidable, 2) both theories are elementarily equivalent.

Definition 7.3. Theory $\mathcal{A}r = \langle \mathcal{L}, \mathcal{C}, AxP \rangle$ is a system of three elements :

$\mathcal{L}$ is a language of first-order. The alphabet of this language contains the set V of variables, symbols of functors : $0, +$, symbol of equality predicate $=$.
The set of well formed-expressions is the union of set of terms T and set of formulas F. The set of terms T is the least set of expressions that contains the set of variables V and the expression 0 and closed with respect to the following two rules: 1) if two expressions τ_1 and τ_2 are terms, then the expression $(\tau_1 + \tau_2)$ is also a term, 2) if the expression τ is a term, then the expression $S(\tau)$ is also a term.

$\mathcal{C}$ is the consequence operation determined by the axioms of predicate calculus and inference rules of first-order logic

AxP The set of axioms of the Ar theory is listed below.

$$\forall_x\ x + 1 \neq 0 \tag{A}$$
$$\forall_x\ x \neq 0 \implies \exists_y x = y + 1 \tag{B}$$
$$\forall_{x,y}\ x + y = y + x \tag{C}$$
$$\forall_{x,y,z}\ x + (y + z) = (x + y) + z \tag{D}$$
$$\forall_{x,y,z}\ x + z = y + z \implies x = y \tag{E}$$
$$\forall_x\ x + 0 = x \tag{F}$$
$$\forall_{x,z}\ \exists_y\ (x = y + z \vee z = y + x) \tag{G}$$
$$\forall_x\ \exists_y\ (x = y + y \vee x = y + y + 1) \tag{H2}$$
$$\forall_x\ \exists_y\ (x = y + y + y \vee x = y + y + y + 1 \vee x = y + y + y + 1 + 1) \tag{H3}$$

. . . . . .

$$\forall_x\ \exists_y \left(\begin{array}{l} x = \underbrace{y + y + \cdots + y}_{k} \vee \\ x = \underbrace{y + y + \cdots + y}_{k} + 1 \vee \\ x = \underbrace{y + y + \cdots + y}_{k} + \underbrace{1 + 1}_{2} \vee \\ \cdots \\ x = \underbrace{y + y + \cdots + y}_{k} + \underbrace{1 + 1 + \cdots + 1}_{k-2} \vee \\ x = \underbrace{y + y + \cdots + y}_{k} + \underbrace{1 + 1 + \cdots + 1}_{k-1} \end{array} \right) \tag{Hk}$$

. . .

The axioms $H2$ -Hk ... may be given a shorter form. Let us introduce numerals, ie. the constants representing term of the form

$\underline{\mathbf{2}} \stackrel{df}{=} 1+1$

$\underline{\mathbf{3}} \stackrel{df}{=} 1+1+1$

$\ldots$

$\underline{\mathbf{k}} \stackrel{df}{=} \underbrace{1+1+\ldots 1}_{k \text{ times}}$

$\ldots$

Now, the axioms take form

$$\forall_x\ x \bmod \underline{\mathbf{2}} = \underline{\mathbf{0}} \vee x \bmod \underline{\mathbf{2}} = \underline{\mathbf{1}} \tag{H2'}$$

$$\forall_x\ x \bmod \underline{\mathbf{3}} = \underline{\mathbf{0}} \vee x \bmod \underline{\mathbf{3}} = \underline{\mathbf{1}} \vee x \bmod \underline{\mathbf{3}} = \underline{\mathbf{2}} \tag{H3'}$$

$\ldots$

$$\forall_x \bigvee_{j=0}^{k-1} x \bmod \underline{\mathbf{k}} = \underline{\mathbf{j}} \tag{Hk'}$$

Let us recall a couple of useful theorems

F1. Theory $\mathcal{T}$ is elementarily equivalent to the theory Ar.[Pre29, Sta84]

F2. Theory Ar is decidable. [Pre29].

F3. The computational complexity of theory Ar, is double exponential $O(2^{2^n})$ this result belongs to Fisher and Rabin, see [FR79].

F4. Theories $\mathcal{T}$ and Ar have non-standard model, see section 7.1, p. 20.

Now, we shall prove a couple of useful theorems of theory $\mathcal{T}$.

First, we shall show that the sentence $\forall_n \exists_{x,y,z}\, n \cdot 3^x + y = 2^z$ is a theorem of the theory $\mathcal{T}$ of addition. Operations of multiplication and power are inaccessible in the theory $\mathcal{T}$. However, we do not need them. We enrich the theory $\mathcal{T}$ adding two functions $P2(\cdot)$ and $P3(\cdot\cdot)$. defined in this way

$P2(0) = 1$	$P3(y,0) = y$
$P2(x+1) = P2(x) + P2(x)$	$P3(y,x+1) = P3(y,x) + P3(y,x) + P3(y,x)$

Lemma 7.3. The definitions given above are correct, i.e. the following sentences aretheorems of the theory with two definitions

$$\mathcal{T} \vdash \forall_x \exists_y\, P2(x) = y \text{ and}$$
$$\mathcal{T} \vdash \forall_{x,y,z} P2(x) = y \wedge P2(x) = z \implies y = z.$$

Similarly, the sentences $\forall_{y,x}\exists_z\, P3(y,x) = z$ and $\forall_{y,x,z,u} P3(y,x) = z \wedge P3(y,x) = u \implies z = u$ are theorems of theory $\mathcal{T}$.

An easy proof goes by induction with respect to the value of variable x.

In the proof of the lemma 7.4 , below, we shall use the definition of the order relation

$$a < b \stackrel{df}{=} \exists_{c\neq 0}\, a + c = b.$$

Making use of the definition of function $P2$ and $P3$ we shall write the formula $P3(n,x) + y = P2(z)$ as it exppresses the same content as expression $n \cdot 3^x + y = 2^z$.

Lemma 7.4. The following sentence is a theorem of the theory $\mathcal{T}$ enriched by the definitions of $P2$ and $P3$ functions.

$$\forall_n \exists_{x,y,z} P3(n,x) + y = P2(z)$$

Proof:
We begin proving by induction that $\mathcal{T} \vdash \forall_n\, n < 2^n$. Namely, we areto prove $\mathcal{T} \vdash \forall_n\, n < P2(n)$. It is easy to see that $\mathcal{T} \vdash 0 < P2(0)$. We shall prove that $\mathcal{T} \vdash \forall_n (n < P2(n) \implies (n+1 < P2(n+1))$. Inequality $n + 1 < P2(n + 1)$ follows from the two given below $\mathcal{T} \vdash n < P2(n)$ and $\mathcal{T} \vdash 1 < P2(n)$. Hence the formula $n + 1 < P2(n) + P2(n))$ is a theorem of theory $\mathcal{T}$. By definition $P2(n) + P2(n) = P2(n + 1)$.
In the similar manner, we can prove the formula $\mathcal{T} \vdash \forall_n\, \forall_x\, P3(n,x) < P2(n + x + x)$
As a consequence we have $\mathcal{T} \vdash \forall_n \exists_{x,y,z}\, P3(n,x) + y = P2(z)$. □

Lemma 7.5. Let $\mathfrak{M}$ be any model of Presburger arithmetic. An element n is reachable iff there exists a triple $\langle x, y, z\rangle$ of reachable elements such that it satisfies the equation $P3(n,x) + y = P2(z)$ i.e. $n \cdot 3^x + y = 2^z$ and elements x, y, z are reachable.

Proof:
If the following formulas are valid in the structure $\mathfrak{M}$
$\{q := 0;\ while\ q \neq x\ do\ q := q + 1\ od\}(x = q)$,
$\{q := 0;\ while\ q \neq y\ do\ q := q + 1\ od\}(y = q)$,
$\{q := 0;\ while\ q \neq z\ do\ q := q + 1\ od\}(z = q)$ and the following equatuin is valid too $P3(n,x) + y = P2(z)$ then it is easy to verify that the formula $\{t := 0;\ while\ n \neq t\ do\ t := t + 1\ od\}(t = n)$ is valid too.

Nr	Argument
1	a1=P2(z) is reachable
2	y+a2 =a1, a2 is reachable and a2=2^z-y
3	a3=P3(1,x) is reachable , a3=3^x
4	{ q:=1; a5:=a3; while a5≠ a2 do q :=q+1; a5:=a5+a3 od } (q* a3=a2) hence q=n

□

A similar fact will be used in the proof of lemma 5.5.

Lemma 7.6. For every element n, if elements x, y, z satisfy the equation $n \cdot 3^x + y = 2^z$ and they are unreachable then the element n is unreachable too.

Proof:
It is easy to observe that $x < z$ and $y < 2^z$. Hence there exists an unreachable element $a \neq 0$ such that $y + a = 2^z$, where a is the difference. We define auxiliary function $D3(m, n)$ by the following recurrence

$$D3(p,q) = \begin{cases} p & \text{when } q = 0 \\ D3(p, q-1) \div 3 & \text{in the opposite case.} \end{cases}$$

It is evident that $n = D3(a, 3^x)$. Since a and 3^x are unreachable then n is also unreachable. □

7.3. An introduction to calculus of programs $\mathcal{AL}$

For the convenience of reader we cite the axioms and inference rules of algorithmic logic.
Note. *Every axiom of algorihmic logic is a tautology.*
Every inference rule of AL is sound. [MS87]

Axioms

axioms of propositional calculus

Ax_1 $((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \delta) \Rightarrow (\alpha \Rightarrow \delta)))$

Ax_2 $(\alpha \Rightarrow (\alpha \vee \beta))$

Ax_3 $(\beta \Rightarrow (\alpha \vee \beta))$

Ax_4 $((\alpha \Rightarrow \delta) \Rightarrow ((\beta \Rightarrow \delta) \Rightarrow ((\alpha \vee \beta) \Rightarrow \delta)))$

Ax_5 $((\alpha \wedge \beta) \Rightarrow \alpha)$

Ax_6 $((\alpha \wedge \beta) \Rightarrow \beta)$

Ax_7 $((\delta \Rightarrow \alpha) \Rightarrow ((\delta \Rightarrow \beta) \Rightarrow (\delta \Rightarrow (\alpha \wedge \beta))))$

Ax_8 $((\alpha \Rightarrow (\beta \Rightarrow \delta)) \Leftrightarrow ((\alpha \wedge \beta) \Rightarrow \delta))$

Ax_9 $((\alpha \wedge \neg\alpha) \Rightarrow \beta)$

Ax_{10} $((\alpha \Rightarrow (\alpha \wedge \neg\alpha)) \Rightarrow \neg\alpha)$

Ax_{11} $(\alpha \vee \neg\alpha)$

axioms of predicate calculus

Ax_{12} $((\forall x)\alpha(x) \Rightarrow \alpha(x/\tau)))$
where term τ is of the same type as the variable x

Ax_{13} $(\forall x)\alpha(x) \Leftrightarrow \neg(\exists x)\neg\alpha(x)$

axioms of calculus of programs

Ax_{14} $K((\exists x)\alpha(x)) \Leftrightarrow (\exists y)(K\alpha(x/y))$ for $y \notin V(K)$

Ax_{15} $K(\alpha \vee \beta) \Leftrightarrow ((K\alpha) \vee (K\beta))$

Ax_{16} $K(\alpha \wedge \beta) \Leftrightarrow ((K\alpha) \wedge (K\beta))$

Ax_{17} $K(\neg\alpha) \Rightarrow \neg(K\alpha)$

Ax_{18} $((x := \tau)\gamma \Leftrightarrow (\gamma(x/\tau) \wedge (x := \tau)true)) \wedge ((q := \gamma\prime)\gamma \Leftrightarrow \gamma(q/\gamma\prime))$

Ax_{19} **begin** $K; M$ **end** $\alpha \Leftrightarrow K(M\alpha)$

Ax_{20} **if** γ **then** K **else** M **fi** $\alpha \Leftrightarrow ((\neg\gamma \wedge M\alpha) \vee (\gamma \wedge K\alpha))$

Ax_{21} **while** γ **do** K **od** $\alpha \Leftrightarrow ((\neg\gamma \wedge \alpha) \vee (\gamma \wedge K(\textbf{while}\ \gamma\ \textbf{do}\ K\ \textbf{od}(\neg\gamma \wedge \alpha))))$

Ax_{22} $\bigcap K\alpha \Leftrightarrow (\alpha \wedge (K \bigcap K\alpha))$

Ax_{23} $\bigcup K\alpha \equiv (\alpha \vee (K \bigcup K\alpha))$

Inference rules

propositional calculus

R_1 $\dfrac{\alpha, (\alpha \Rightarrow \beta)}{\beta}$ (also known as modus ponens)

predicate calculus

R_6 $\dfrac{(\alpha(x) \Rightarrow \beta)}{((\exists x)\alpha(x) \Rightarrow \beta)}$

R_7 $\dfrac{(\beta \Rightarrow \alpha(x))}{(\beta \Rightarrow (\forall x)\alpha(x))}$

calculus of programs AL

R_2 $\dfrac{(\alpha \Rightarrow \beta)}{(K\alpha \Rightarrow K\beta)}$

R_3 $\dfrac{\{s(\textbf{if}\ \gamma\ \textbf{then}\ K\ \textbf{fi})^i(\neg\gamma \wedge \alpha) \Rightarrow \beta\}_{i\in N}}{(s(\textbf{while}\ \gamma\ \textbf{do}\ K\ \textbf{od}\ \alpha) \Rightarrow \beta)}$

R_4 $\dfrac{\{(K^i\alpha \Rightarrow \beta)\}_{i\in N}}{(\bigcup K\alpha \Rightarrow \beta)}$

R_5 $\dfrac{\{(\alpha \Rightarrow K^i\beta)\}_{i\in N}}{(\alpha \Rightarrow \bigcap K\beta)}$

In rules R_6 and R_7, it is assumed that x is a variable which is not free in β, i.e. $x \notin FV(\beta)$. The rules are known as the rule for introducing an existential quantifier into the antecedent of an implication and the rule for introducing a universal quantifier into the successor of an implication. The rules R_4 and R_5 are algorithmic counterparts of rules R_6 and R_7. They are of a different character, however, since their sets of premises are infinite. The rule R_3 for introducing a **while** into the antecedent of an implication of a similar nature. These three rules are called ω-rules. The rule R_1 is known as *modus ponens*, or the *cut*-rule. In all the above schemes of axioms and inference rules, α, β, δ are arbitrary formulas, γ and γ' are arbitrary open formulas, τ is an arbitrary term, s is a finite sequence of assignment instructions, and K and M are arbitrary programs.

Theorem 7.1. (*meta*-theorem on completeness of the calculus of programs $\mathcal{AL}$)
Let $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A}x \rangle$ be a consistent theory, let $\alpha \in \mathcal{L}$ be a formula. The following conditions are equivalent

(*i*) Formula α is a theorem of the theory T, $\alpha \in \mathcal{C}(\mathcal{A}x)$,

(*ii*) Formula α is valid in every model of the theory T, $\mathcal{A}x \models \alpha$.

The proof may be found in [MS87].

7.4. An introduction to algorithmic theory of natural numbers $\mathcal{ATN}$

The language of algorithmic theory of natural numbers $\mathcal{ATN}$ is very simple. Its alphabet contains one constant 0 *zero* , one one-argument functor s and predicate = of equality. We shall write $x + 1$ instead of $s(x)$. Axioms of $\mathcal{ATN}$ were presented in the book [MS87]

$$\begin{array}{lll} A_1 & \forall x\{q := 0;\ \textbf{while}\ q \neq x\ \textbf{do}\ q := s(q)\ \textbf{od}\}(q = x) & (R) \\ A_2 & \forall x\ s(x) \neq 0 & (N) \\ A_3 & \forall x \forall y\, s(x) = s(y) \implies x = y & (J) \end{array}$$

We can add another two-argument functor + and its definition

$$A_4 \quad \forall x\, \forall y\{q := 0; w := x;\ \textbf{while}\ q \neq y\ \textbf{do}\ q := s(q)\ ; w := s(w)\textbf{od}\}(x + y = w) \qquad (D)$$

The termination property of the program in A_4 is a theorem of $\mathcal{ATN}$ theory as well as the formulas $x + 0 = x$ and $x + s(y) = s(x + y)$.

A sample (12 – 15) of Theorems of $\mathcal{ATN}$

$$\mathcal{ATN} \vdash \exists_x\, \alpha(x) \Leftrightarrow \{x := 0\} \bigcup \{x := x + 1\}\alpha(x) \tag{12}$$

$$\mathcal{ATN} \vdash \forall_x\, \alpha(x) \Leftrightarrow \{x := 0\} \bigcap \{x := x + 1\}\alpha(x) \tag{13}$$

scheme of induction

$$\mathcal{ATN} \vdash \Big(\alpha(x/0) \wedge \forall_x\big(\alpha(x) \Rightarrow \alpha(x/s(x))\big)\Big) \implies \forall_x \alpha(x) \tag{14}$$

Correctness of Euclid's algorithm

$$\mathcal{ATN} \vdash \left(\begin{array}{l} n_0 > 0 \wedge \\ m_0 > 0 \end{array}\right) \implies \left\{\begin{array}{l} n := n_0;\ m := m_0; \\ \textbf{while}\, n \neq m\, \textbf{do} \\ \quad \textbf{if}\, n > m\, \textbf{then}\ n := n \dot{-} m \\ \qquad \textbf{else}\ m := m \dot{-} n \\ \quad \textbf{fi} \\ \textbf{od} \end{array}\right\} (n = gcd(n_0, m_0)) \tag{15}$$

The theory $\mathcal{ATN}$ enjoys an important propety of categoricity.

Theorem 7.2. (*meta*-theorem on categoricity of $\mathcal{ATN}$)
Every model $\mathfrak{A}$ of the algorithmic theory of natural numbers is isomorphic to the structure $\mathfrak{N}$, c.f. subsection 7.1.

7.5. Proof of lemma 4.1

Let P and P' be two programs. Let α be any formula. The semantic property *programs P and P' are equivalent with respect to the postcondition α* is expressed by the formula of the form $(\{P\}\,\alpha \Leftrightarrow \{P'\}\,\alpha)$.
We shall use the following tautology of calculus of programs $\mathcal{AL}$.

$$\vdash \left(\overbrace{\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{if } \delta \textbf{ then } K \textbf{ else } M \textbf{ fi} \\ \textbf{od}; \end{array} \right\}}^{P:} \alpha \Leftrightarrow \overbrace{\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od}; \\ \quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od} \\ \textbf{od} \end{array} \right\}}^{P':} \alpha \right) \tag{16}$$

We apply the axioms Ax20 and Ax21

$$\vdash \left(\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{if } \delta \textbf{ then } K \textbf{ else } M \textbf{ fi} \\ \textbf{od}; \end{array} \right\} \alpha \Leftrightarrow \left\{ \begin{array}{l} \textbf{if } \gamma \textbf{ then} \\ \quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od}; \\ \quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od} ; \\ \quad \textbf{while } \gamma \textbf{ do} \\ \quad\quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od}; \\ \quad\quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od} \\ \quad \textbf{od} \\ \textbf{fi} \end{array} \right\} \alpha \right) \tag{17}$$

We can omit the instruction **if** (why?) . We swap internal instructions **while** inside the instruction **while**.

$$\vdash \left(\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{if } \delta \textbf{ then } K \textbf{ else } M \textbf{ fi} \\ \textbf{od}; \end{array} \right\} \alpha \Leftrightarrow \left\{ \begin{array}{l} \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od} ; \\ \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od} \\ \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od}; \\ \quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od} \\ \textbf{od} \end{array} \right\} \alpha \right) \tag{18}$$

We can safely skip the second instruction while.

$$\vdash \left(\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{if } \delta \textbf{ then } K \textbf{ else } M \textbf{ fi} \\ \textbf{od}; \end{array} \right\} \alpha \Leftrightarrow \left\{ \begin{array}{l} \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od} ; \\ \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od}; \\ \quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od} \\ \textbf{od} \end{array} \right\} \alpha \right) \tag{19}$$

Now, we put $(\gamma \stackrel{df}{=} n \neq 1)$, $(\delta \stackrel{df}{=} even(n))$, $(K \stackrel{df}{=} n \div 2)$ and $(M \stackrel{df}{=} 3n+1)$. We make use of two theorems $(even(n) \implies n \neq 1)$ and $(M\delta)$ of theory $\mathcal{T}$ c.f. subsection 7.2 (hence, they are valid in the structure of natural numbers).

$$\mathcal{T} \vdash \left(\left\{ \begin{array}{l} \textbf{while } n \neq 1 \textbf{ do} \\ \quad \boxed{\begin{array}{l} \textbf{if } even(n) \\ \textbf{then } n := n \div 2 \\ \textbf{else } n := 3n+1 \\ \textbf{fi} \end{array}} \\ \textbf{od}; \end{array} \right\} (n=1) \Leftrightarrow \left\{ \begin{array}{l} \textbf{while } even(n) \textbf{ do } n := n \div 2 \textbf{ od}; \\ \textbf{while } n \neq 1 \textbf{ do} \\ \quad \boxed{\begin{array}{l} n := 3n+1; \\ \textbf{while } even(n) \textbf{ do } n := n \div 2 \textbf{ od} \end{array}} \\ \textbf{od} \end{array} \right\} (n=1) \right). \tag{20}$$

This ends the proof.

□

7.6. Proof of invariant of algorithm $Gr3$

We are going to prove the following implication (22) is a theorem of algorithmic theory of natural numbers $\mathcal{ATN}$.

$$\mathcal{ATN} \vdash (\varphi \implies \{\Delta_3\}\varphi) \tag{21}$$

Symbols φ and Δ_3 are explained below.

$$\overbrace{\left(\begin{array}{l} n\cdot 3^i + Y_i = m_i\cdot 2^{Z_i}\wedge \\ Z_i = \sum\limits_{j=0}^{i} k_j \wedge X_i = i\wedge \\ Y_i = \sum\limits_{j=0}^{i-1}\left(3^{i-1-j}\cdot 2^{Z_j}\right)\wedge \\ m_0 = n \wedge\ k_0 = exp(m_0,2)\wedge \\ \bigwedge\limits_{0<l\le i}\big(m_l = m_{l-1}/2^{k_l}\wedge \\ k_l = exp(m_l,2)\big) \end{array}\right)}^{\varphi} \implies \overbrace{\left\{\begin{array}{l} aux := 3*m_i+1; \\ k_{i+1} := exp(aux,2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ Y_{i+1} := 3Y_i + 2^{Z_i}; \\ Z_{i+1} := Z_i + k_{i+1}; \\ X_i := i;; \\ \hline i := i+1; \end{array}\right\}}^{\Delta_3} \overbrace{\left(\begin{array}{l} n\cdot 3^i + Y_i = m_i\cdot 2^{Z_i}\wedge \\ Z_i = \sum\limits_{j=0}^{i} k_j \wedge\wedge X_i = i \\ Y_i = \sum\limits_{j=0}^{i-1}\left(3^{i-1-j}\cdot 2^{Z_j}\right)\wedge \\ m_0 = n \wedge\ k_0 = exp(m_0,2)\wedge \\ \bigwedge\limits_{0<l\le i}\big(m_l = m_{l-1}/2^{k_l}\wedge \\ k_l = exp(m_l,2)\big) \end{array}\right)}^{\varphi} \tag{22}$$

We apply the axiom of assignment instruction Ax_{18}. Note, we applied also axiom Ax_{19} of composed instruction.

Namely, in the implication (22) we replace the its successor $\{\Delta_3\}\varphi$ by the equivalent formula $\{\Delta_3^{(1)}\}\varphi^{(1)}$.

$$\varphi \implies \overbrace{\left\{\begin{array}{l} aux := 3*m_i+1; \\ k_{i+1} := exp(aux,2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ Y_{i+1} := 3Y_i + 2^{Z_i}; \\ Z_{i+1} := Z_i + k_{i+1}; \\ \hline X_i := i;; \end{array}\right\}}^{\Delta_3^{(1)}} \overbrace{\left(\begin{array}{l} n\cdot 3^{(i+1)} + Y_{(i+1)} = m_{(i+1)}\cdot 2^{Z_{(i+1)}}\wedge \\ Z_{(i+1)} = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1)\wedge \\ Y_{(i+1)} = \sum\limits_{j=0}^{i}\left(3^{(i+1)-1-j}\cdot 2^{Z_j}\right)\wedge \\ m_0 = n \wedge\ k_0 = exp(m_0,2)\wedge \\ \bigwedge\limits_{0<l\le(i+1)}\big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l,2)\big) \end{array}\right)}^{\varphi^{(1)}} \tag{23}$$

$$\varphi \implies \left\{\begin{array}{l} aux := 3*m_i+1; \\ k_{i+1} := exp(aux,2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ Y_{i+1} := 3Y_i + 2^{Z_i}; \\ \hline Z_{i+1} := Z_i + k_{i+1}; \end{array}\right\} \left(\begin{array}{l} n\cdot 3^{(i+1)} + Y_{(i+1)} = m_{(i+1)}\cdot 2^{Z_{(i+1)}}\wedge \\ Z_{(i+1)} = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1)\wedge \\ Y_{(i+1)} = \sum\limits_{j=0}^{i}\left(3^{(i+1)-1-j}\cdot 2^{Z_j}\right)\wedge \\ m_0 = n \wedge\ k_0 = exp(m_0,2)\wedge \\ \bigwedge\limits_{0<l\le(i+1)}\big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l,2)\big)\wedge \end{array}\right) \tag{24}$$

$$\varphi \implies \left\{\begin{array}{l} aux := 3*m_i+1; \\ k_{i+1} := exp(aux,2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ \hline Y_{i+1} := 3Y_i + 2^{Z_i}; \end{array}\right\} \left(\begin{array}{l} n\cdot 3^{(i+1)} + Y_{(i+1)} = m_{(i+1)}\cdot 2^{(Z_i+k_{i+1})}\wedge \\ (Z_i + k_{i+1}) = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1)\wedge \\ Y_{(i+1)} = \sum\limits_{j=0}^{i}\left(3^{(i+1)-1-j}\cdot 2^{Z_j}\right)\wedge \\ m_0 = n \wedge\ k_0 = exp(m_0,2)\wedge \\ \bigwedge\limits_{0<l\le(i+1)}\big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l,2)\big) \end{array}\right) \tag{25}$$

$$\varphi \implies \left\{ \begin{array}{c} aux := 3 * m_i + 1; \\ \hline k_{i+1} := exp(aux, 2); \\ m_{i+1} := aux/2^{k_{i+1}}; \end{array} \right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = m_{(i+1)} \cdot 2^{(Z_i + k_{i+1})} \wedge \\ (Z_i + k_{i+1}) = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum\limits_{j=0}^{i} \left(3^{(i+1)-1-j} \cdot 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \bigwedge\limits_{0<l\leq(i+1)} \left(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \right) \end{array} \right) \tag{26}$$

$$\varphi \implies \left\{ \begin{array}{c} aux := 3 * m_i + 1; \\ \hline k_{i+1} := exp(aux, 2); \end{array} \right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = (aux/2^{k_{i+1}}) \cdot 2^{(Z_i + k_{i+1})} \wedge \\ (Z_i + k_{i+1}) = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum\limits_{j=0}^{i} \left(3^{(i+1)-1-j} \cdot 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \bigwedge\limits_{0<l\leq(i+1)} \left(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \right) \end{array} \right) \tag{27}$$

$$\varphi \implies \left\{ \ aux := 3 * m_i + 1; \ \right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = (aux/2^{(exp(aux,2))}) \cdot 2^{(Z_i + (exp(aux,2)))} \wedge \\ (Z_i + (exp(aux, 2))) = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum\limits_{j=0}^{i} \left(3^{(i+1)-1-j} \cdot 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \bigwedge\limits_{0<l\leq(i+1)} \left(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \right) \end{array} \right) \tag{28}$$

$$\varphi \implies \{\} \overbrace{\left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = ((3 * m_i + 1)/2^{(exp((3*m_i+1),2))}) \cdot 2^{(Z_i + (exp((3*m_i+1),2)))} \wedge \\ (Z_i + (exp((3 * m_i + 1), 2))) = \sum\limits_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum\limits_{j=0}^{i} \left(3^{(i+1)-1-j} \cdot 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \bigwedge\limits_{0<l\leq(i+1)} \left(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \right) \end{array} \right)}^{\psi} \tag{29}$$

The implications (22) –(29) are mutually equivalent.
One can easily verify that the last implication $(\varphi \implies \psi)$ (29) is a theorem of Presburger arithmetic $\mathcal{T}$ and hence it is a theorem of $\mathcal{ATN}$ theory.
Therefore the first implication $(\varphi \implies \{\Delta_3\}\varphi)$ (22) is a theorem of algorithmic theory of natural numbers $\mathcal{ATN}$.

q.e.d.

Remark 7.3. Note, the process of creation a proof like this can be automatized.
The verification of the above proof can be automatized too.

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