Collatz conjecture becomes theorem

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1 Introduction

The $3x + 1$ problem remained open for over 80 years. It has been noticed in 1937 by Lothar Collatz. The problem became quite popular due to its wording, for it is short and easy to comprehend. Collatz remarked that for any given natural number $n > 0$, the sequence $\{n_i\}$ defined by the following recurrence

$$\left.\begin{aligned} &n_0 = n \\ &n_{i+1} = \begin{cases} n_i \div 2 & \text{when } n_i \text{ is even} \\ 3 \cdot n_i + 1 & \text{when } n_i \text{ is odd} \end{cases} \quad \text{for } \ i \geq 0 \end{aligned}\right\} \qquad \text{(rec1)}$$

seem always reach the value 1.

He formulated the following conjecture

$$\text{for all } n \text{ exists } i \text{ such that } n_i = 1 \qquad \text{(Collatz conjecture)}$$

One can give another formulation of the hypothesis of Collatz [1]. The number of papers devoted to the problem surpasses 200, c.f. [Lag10] . It is worthwhile to consult social media: wikipedia, youtube

[1] Let $f(n,0) \overset{df}{=} n$, and $f(n,i+1) \overset{df}{=} \begin{cases} f(n,i)/2 & \text{if } f(n,i) \text{ is even} \\ 3 \cdot f(n,i)+1 & \text{if } f(n,i) \text{ is odd} \end{cases}$. Now, conjecture reads $\forall_n \, \exists_i \, f(n,i) = 1$.

etc, there you can find some surprising ideas to prove the Collatz hypothesis as well as a technical analysis of the problem.
Computers are used and are still crunching numbers in the search of an eventual counterexample to the Collatz conjecture. The reports on progress appear each year. We claim that the counterexample approach is pointless, i.e. the computers can be turned off. Namely, we shall prove that the program that searches a counterexample will never stop.
Our goal will be achieved if we prove that for each number n the computation of the following Cl algorithm is finite.

$$Cl : \left\{ \begin{array}{l} \textbf{while } n \neq 1 \textbf{ do} \\ \quad \textbf{if } even(n) \textbf{ then } n := n \div 2 \textbf{ else } n := 3n+1 \textbf{ fi} \\ \textbf{od} \end{array} \right\}$$

One needs

- a formula Θ_{Cl} such that it expresses the termination property of program Cl,
- a verifiable proof Π of the formula and
- a definition of relation $\mathcal{C}$ of deducibility.

Ah, we need also a specification of the domain in which the algorithm is to be executed, i.e. the axioms $\mathcal{A}x$ of the algebraic structure of natural numbers.

QUESTION 1. How to express the termination property of a program K as a formula Θ_K (i.e. a Boolean expression)?

Note, there is no a universal algorithm that expresses the halting property of a given program K by an appropriate first-order logical formula Θ_K. First, let us recall the theorem on incompleteness of arithmetics, cf. Kurt Gödel . According to Gödel, the property *to be a natural number* is not expressible by any set of first-order formulas. The reader may wish to note, that halting property of the algoritm

$$q := 0; \textbf{while } q \neq n \textbf{ do } q := q+1 \textbf{ od}$$

is valid in a data structure iff n is a standard (i.e. reachable) natural number. Therefore the halting property allow to define the set of natural numbers. In this situation it seems natural to pass from first-order language to another more expressive language. There are three candidates: 1°a second-order language admitting variables and quantifiers over sets, 2°the language of infinte disjunctions and conjunctions $\mathcal{L}_{\omega_1\omega}$ and 3°language of algorithmic logic. Problem with second order logic is in lack of adequate definition of consequence operation. True, we can limit our considerations to the case of finite sets (*aka*, weak second order logic). Still we do not know a complete set of axioms and inference rules for the weak second-order logic. Applying second-order logic to program analysis results in a heavy overhead. Because first you have to translate the semantic property of the program into a property of a certain sequence or set. prove this property and make a backward translation. The question of whether this approach is acceptable to software engineers seems to be appropriate.
The language of infinite disjunctions and conjunctions is not an acceptable tool for software engeeners.

We shall use the language and the consequence operation offered by algorithmic logic i.e. calculus of programs. We enlarge the set of well formed expressions: beside terms and formulas of first order language we accept algorithms and we modify the definition of logical formulas. The simplest algorithmic formulas are of the form: $\langle algorithm \rangle \langle formula \rangle$.
As an example of an algorithmic formula consider the expression

$$\forall_n \{q := 0; \mathbf{while}\ q \neq n\ \mathbf{do}\ q := q + 1\ \mathbf{od}\}\ (n = q)$$

The latter formula is valid iff every element n can be reached from 0 by adding 1.

Now our goal is to prove the following formula

$$\forall_n \left\{ \begin{array}{l} \textbf{while } n \neq 1 \textbf{ do} \\ \quad \textbf{if } even(n) \textbf{ then } n := n \div 2 \\ \quad \textbf{else } n := 3n + 1 \textbf{ fi} \\ \textbf{od} \end{array} \right\} (n = 1) \tag{1}$$

from the axioms of algorithmic theory of natural numbers $\mathcal{ATN}$, c.f. subsection 6.4. For the formula (1) expresses the termination property of program Cl.

Question 2. How to prove such algorithmic formula?
Note, all structures that assure the validity of axioms of the $\mathcal{ATN}$ theory are isomorphic (this is the categoricity *meta*-theorem). Therefore, the termination formula, can be either proved (with the inference rules and axioms of calculus of programs $\mathcal{AL}$, or validated in this unique model of axioms of $\mathcal{ATN}$.

Let us make a simple observation. The computation of Collatz algorithm if succesful goes through intermediate values. The following diagram (Dg) illustrates a computation where all odd numbers were exposed as stacked fractions.

$$n \to \frac{n}{2^{k_0}} \to \frac{3*(\frac{n}{2^{k_0}})+1}{2^{k_1}} \to \frac{3*\frac{\frac{n}{2^{k_0}}\cdot 3+1}{2^{k_1}}+1}{2^{k_2}} \cdots \to \frac{3\cdot\frac{3\cdot\frac{3\cdot\frac{3\cdot\frac{3\cdot\frac{n}{2^{k_0}}+1}{2^{k_1}}+1}{2^{k_2}}+1}{(\dots)}+1}{2^{k_{x-1}}}+1}{2^{k_x}} = 1 \tag{Dg}$$

where $k_0 = expo(n, 2), k_1 = expo(3 \cdot \frac{n}{2^{k_0}} + 1, 2), k_2 = expo(3 * \frac{(3\cdot\frac{n}{2^{k_0}}+1)}{2^{k_1}} + 1, 2), \dots$ [2]. In our earlier paper [MS21] we studied the halting formula of the Collatz algorithm. We remarked that the computation of Collatz algorithm is finite iff there exist three natural numbers x, y, z such that:
a) the equation $n \cdot 3^x + y = 2^z$ is satisfied and

[2] **Note**, the function *expo* returns the largest exponent of 2 in the prime factorization of number x .
$expo(x, 2) \stackrel{df}{=} \{l := 0; y := x; \textbf{while } even(y) \textbf{ do } l := l + 1;\ y := y/2 \textbf{ od}\}(\text{result} = l)$ i.e. $l = expo(x, 2)$. We introduce the notation *expo* instead *exp* for our friends programmers who interpret the notation *exp* as exponentiation.

b) the computation of another algorithm IC, cf. page 19, is finite, the algorithm computes on triples$\langle x, y, z \rangle$.
It is worthwhile to mention that the subsequent triples are decreasing during computation.
The proof we wrote there [MS21] is overly complicated.

Here we show that the 4-argument relation

$$\{n, x, y, z\} : \{IC\}(true).$$

is elementary recursive, since it may be expressed by an arihmetic expression with operator $\sum$.
The present paper shows arguments simpler and easier to follow.

2 Collatz tree

Definition 1 *Collatz tree $\mathcal{DC}$ is a subset $D \subset N$ of the set N of natural numbers and the function f defined on the set $D \setminus \{0, 1\}$.*

$$\mathcal{DC} = \langle D, f \rangle$$

where $D \subset N, 1 \in D, f \colon D \setminus \{0, 1\} \to D$.
Function f is determined as follows

$$f(n) = \begin{cases} n \div 2 & \textit{when } n \mod 2 = 0 \\ 3n + 1 & \textit{when } n \mod 2 = 1 \end{cases}$$

, the set D is the least set containing the number 1 and closed with respect to the function f,

$$D = \{n \in N : \exists_{i \in N}\ f^i(n) = 1\ \}\ .$$

As it is easy to see, this definition is highly entangled and the decision whether the set D contains every natural number is equivalent to the Collatz problem.

Remark 2 *Set D has the following properties :*

$$x \in D \implies (x+x) \in D \tag{2}$$

$$\big(x \in D \implies (\exists_y x = y + y \implies y \in D)\big) \tag{3}$$

$$\big(x \in D \implies (\exists_y x = y + y + 1 \implies (x + x + x + 1) \in D)\big) \tag{4}$$

$$\big(x \in D \implies (\exists_e \exists_z e = z + z + 1 \land x = e + e + e + 1 \implies e \in D)\big) \tag{5}$$

Implications (2) and (5) show left and right son of element x.

Conjecture 3 *The Collatz tree contains all the reachable natural numbers.*

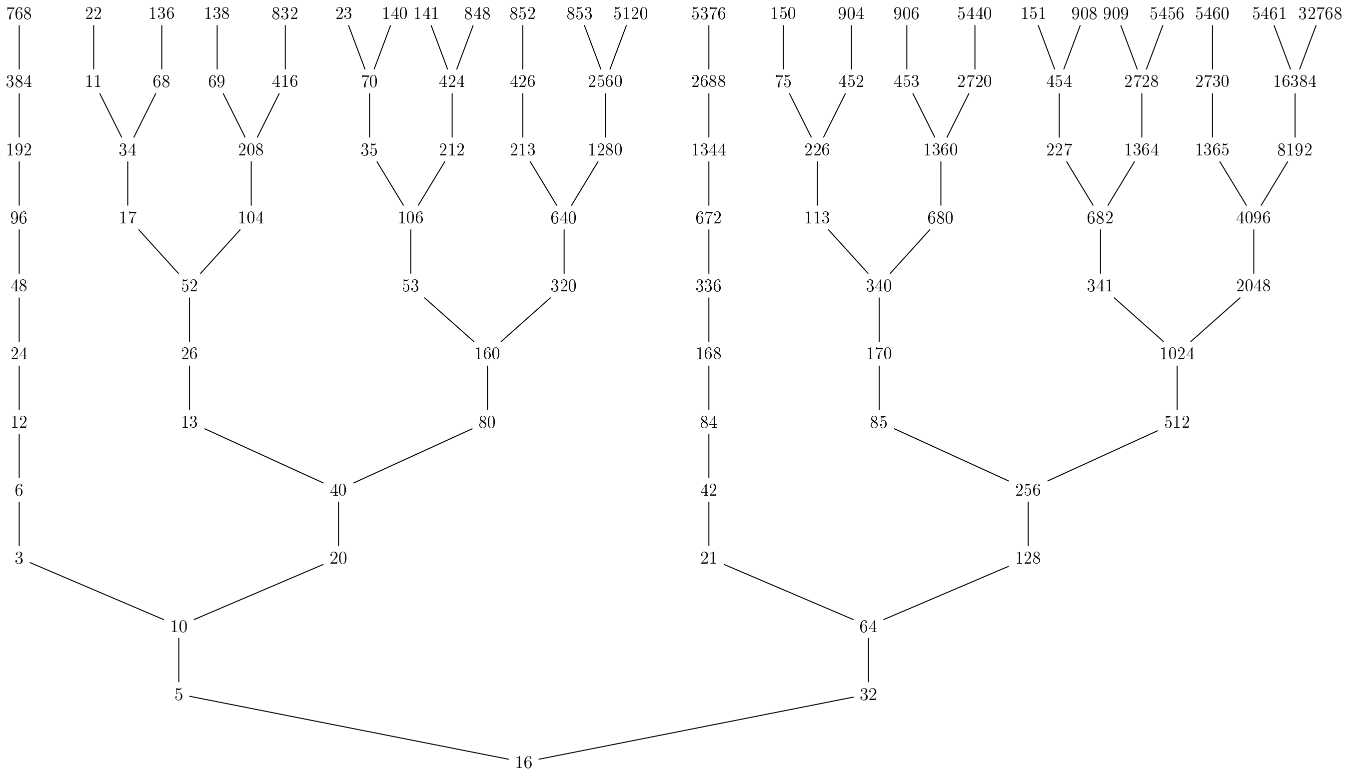


Fig. 1. A fragment of Collatz tree, levels 4-15. It does not include levels 0-3, they consist of elements 1 — 2 — 4 — 8 — .

3 Four algorithms, relatives of Cl algorithm

In this section we present an algorithm Gr equivalent to the algorithm Cl and three algorithms $Gr1, Gr2, Gr3$ that are successive extensions of the Gr algorithm.

Lemma 4 *The following algorithm Gr is equivalent to Collatz algorithm Cl.*

Gr:

```
while even(n) do n:= n ÷ 2 od ;
while n ≠1 do
  n:= 3*n+1;
  while even(n) do n:= n ÷ 2 od
od ;
```

Proof. The equivalence of the algorithms Cl and Gr is intuitive. Compare the recurrence of Collatz (rec1) and the following recurrence (rec2) that is calculated by the algorithm Gr.

$$\left.\begin{array}{rl} k_0 = expo(n,2) \ \wedge \ m_0 = \frac{n}{2^{k_0}} & \\ k_{i+1} = expo(3m_i+1,2) \ \wedge m_{i+1} = \frac{3m_i+1}{2^{k_{i+1}}} & \text{for } i \geq 0 \end{array}\right\} \quad \text{(rec2)}$$

One can say the algorithm Gr is obtained by the elimination of **if** instruction from the Cl algorithm. However, construction of a formal proof is a non-obvious task. A sketch of a proof is given in subsection 6.5 We are encouraging the reader to fill the details. ■

Next, we present the algorithm $Gr1$, an extension of algorithm Gr.

```
var  n,l,i :integer ; k,m :arrayof  integer;
        Γ1: | i := 0; l := 0;
            | while even(n)  do  n:= n ÷ 2;  l := l + 1 od ;  k_i := l;  m_i :=n:
        while n ≠1  do
Gr1:
            | { m_i=n }    n:= 3*n+1;   l := 0;
        Δ1: | while even(n)  do  n:= n ÷2;  l := l + 1od ; k_{i+1} := l;  m_{i+1} :=n;
            | {m_{i+1} = (3·m_i+1)/2^{K_i} ∧ k_{i+1} = exp(3 * m_i + 1, 2)}    i := i + 1
        od
```

Lemma 5 *Algorithm $Gr1$ has the following properties:*
(i) Algorithms Gr and $Gr1$ are equivalent with respect to the halting property.

(ii) The sequences $\{m_i\}$ and $\{k_i\}$ calculated by the algorithm $Gr1$ satisfy the recurrence rec2.

Proof. Both statements are very intuitive. Algorithm $Gr1$ is an extension of algorithm Gr. The inserted instructions do not interfere with the halting property of algorithm $Gr1$. Second part of the lemma follows easily from the remark that $k_0 = exp(\mathrm{n}, 2)$ and $m_0 = \frac{n}{2^{k_0}}$ and that for all $i > 0$ we have $k_{i+1} = exp(3 * m_i + 1, 2)$ and $m_{i+1} = \frac{3 \cdot m_i + 1}{2^{k_{i+1}}}$. ■

Each odd number m in Collatz tree, $m \in D$, initializes a new branch. Let us give a color number $x + 1$ to each new branch emanating from a branch with color number x. Note, for every natural number$p$ the set of branches of the color p is infinite. Let W_x denote the set of natural numbers that obtained the color x.
Besides the levels of Collatz tree, one can distinguish the structure of strata in the tree.

Definition 6 *Inductive definition of strata.*

$$W_0 \stackrel{df}{=} \{n \in N : n = 2^{expo(n,2)}\} \tag{W0}$$

$$W_{x+1} \stackrel{df}{=} \{n \in N : \exists_{m \in W_x} \left(3 \cdot \frac{n}{2^{expo(n,2)}} + 1 = m \right)\} \tag{Wx+1}$$

A couple of observations will be used in the sequel.

Remark 7 *Among properties of sets W_i we find*

(1) *Each set W_i is infinite. If it contains a number n then the number $2n \in W_i$.*

(2) *The set W_0 contains one odd number 1. All other sets contain infinitely many odd numbers. For if $a = 2j+1$ and $a \in W_i$ then $4a+1 \in W_i$.*

(3) *We define the function $f\colon N \to N$ as follow:* $f(n) = \begin{cases} n \div 2 & \text{when } n \text{ is even} \\ 3n+1 & \text{when } n \text{ is odd } n \neq 1 \\ undefined & \text{when } n = 1 \end{cases}$

(4) *Let $x > 0$ be a natural number. Let the sequence $\{o_j\}_{i \in N}$ contain all odd numbers that are in the set W_x. Let $S_{o_j} = \left\{2^i \cdot o_j\right\}$ be the set. Every set W_x may be partitioned as follow $W_x = \bigcup_{j=0}^{\infty} S_{o_j}$. If $j \neq j'$ then $S_{o_j} \cap S_{o_{j'}} = \emptyset$*

(5) *For every $i, j \in Nat$ if $i \neq j$ then $W_i \cap W_j = \emptyset$.*

(6) *For every $n, j \in Nat$ if $n \in W_j \wedge j > 1 \wedge n \mod 2 = 1$ then $3n+1 \in W_{j-1}$.*

(7) *The sequence of sets $\left\{\bigcup_{i=0}^{j} W_i\right\}_{j \in N}$ is monotone, increasing.*

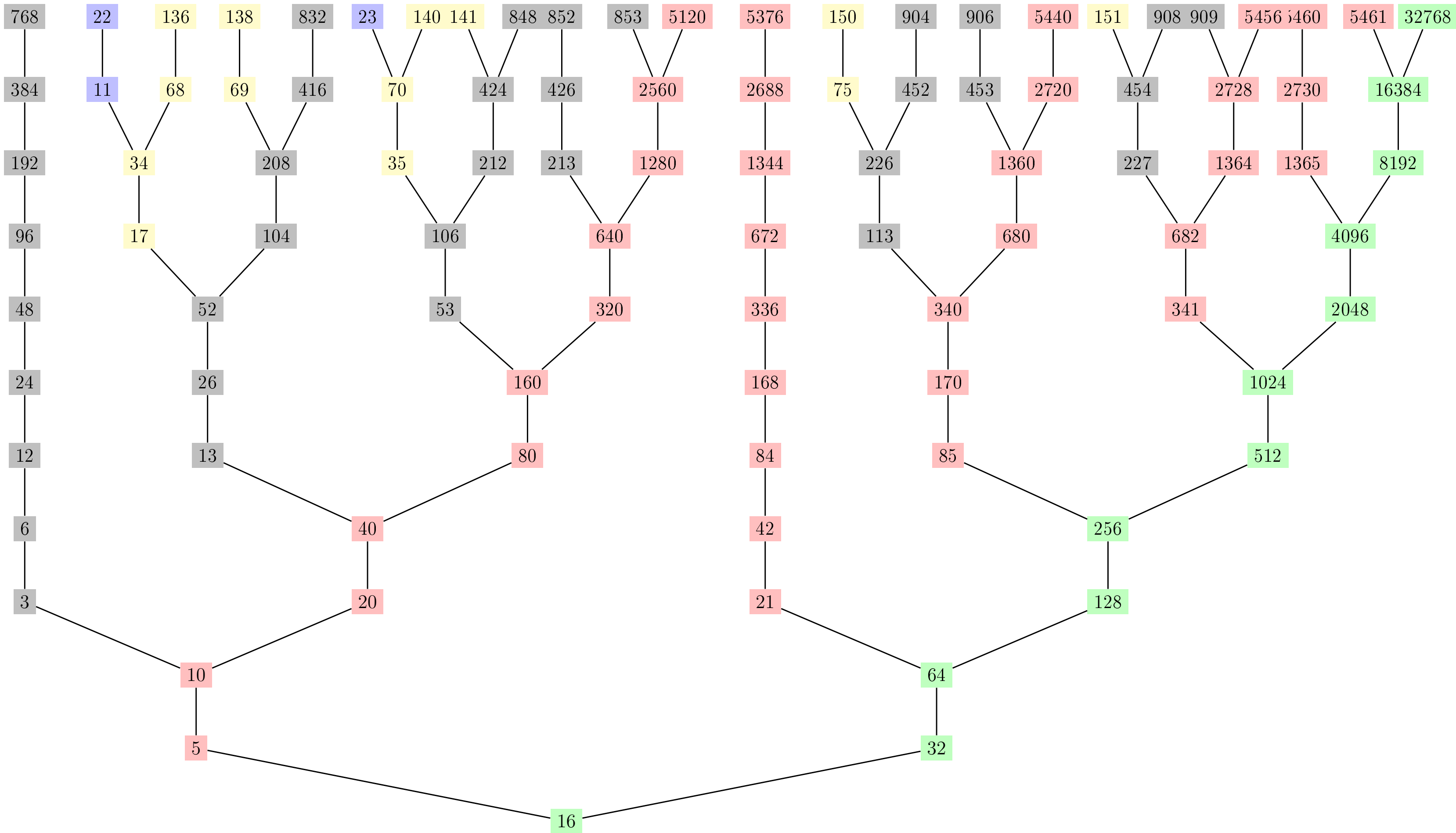


Fig. 2. Strata $W_0 - -W_4$ of Collatz tree

Let s be a variable not occurring in algorithm $Gr1$. The following lemma states the partial correctnes of the algorithm $Gr1$ w.r.t. precondition $s = n$ and postcondition $s \in W_i$.

Lemma 8 *Algorithm $Gr1$ computes the number i of storey W_i of number n,*

$$\{Gr1\}(true) \implies \big((s = n) \implies \{Gr1\}(s \in W_i)\big)$$

Next, we present another algorithm $Gr2$ and a lemma.

```
     var n,l,i,x,y,z :integer ; k,m :arrayof integer;
         Γ2: i := 0; l := 0;
             while even(n) do n:= n ÷ 2; l := l + 1 od ;
             z, k_i := l; m_i:=n; y := 0;
Gr2: while m_i ≠1 do
         Δ2: n:= 3*n+1; i := i + 1; l := 0 ;
             while even(n) do n:= n ÷ 2; l := l + 1 od ;
             k_i := l; m_i:=n; z := z + k_i; y := 3 * y + 2^z; x := i
     od
```

Lemma 9 *Algorithm $Gr2$ has the following properties:*
(i) Both algorithms $Gr1$ and $Gr2$ are equivalent with respect to the halting property.
(ii) Formula φ : $\boxed{n \cdot 3^i + y = m_i \cdot 2^z}$ is an invariant of the program $Gr2$ i.e. the formulas (6) and (7)

$$\{\Gamma_2\}\,(n \cdot 3^i + y = m_i \cdot 2^z) \tag{6}$$

$$(n \cdot 3^i + y = m_i \cdot 2^z) \implies \{\Delta_2\}(n \cdot 3^i + y = m_i \cdot 2^z) \tag{7}$$

are theorems of the algorithmic theory of numbers $\mathcal{ATN}$.

Proof. Proofs of these formulas are easy, it suffices to apply the axiom of assignment instruction Ax_{18}, ■

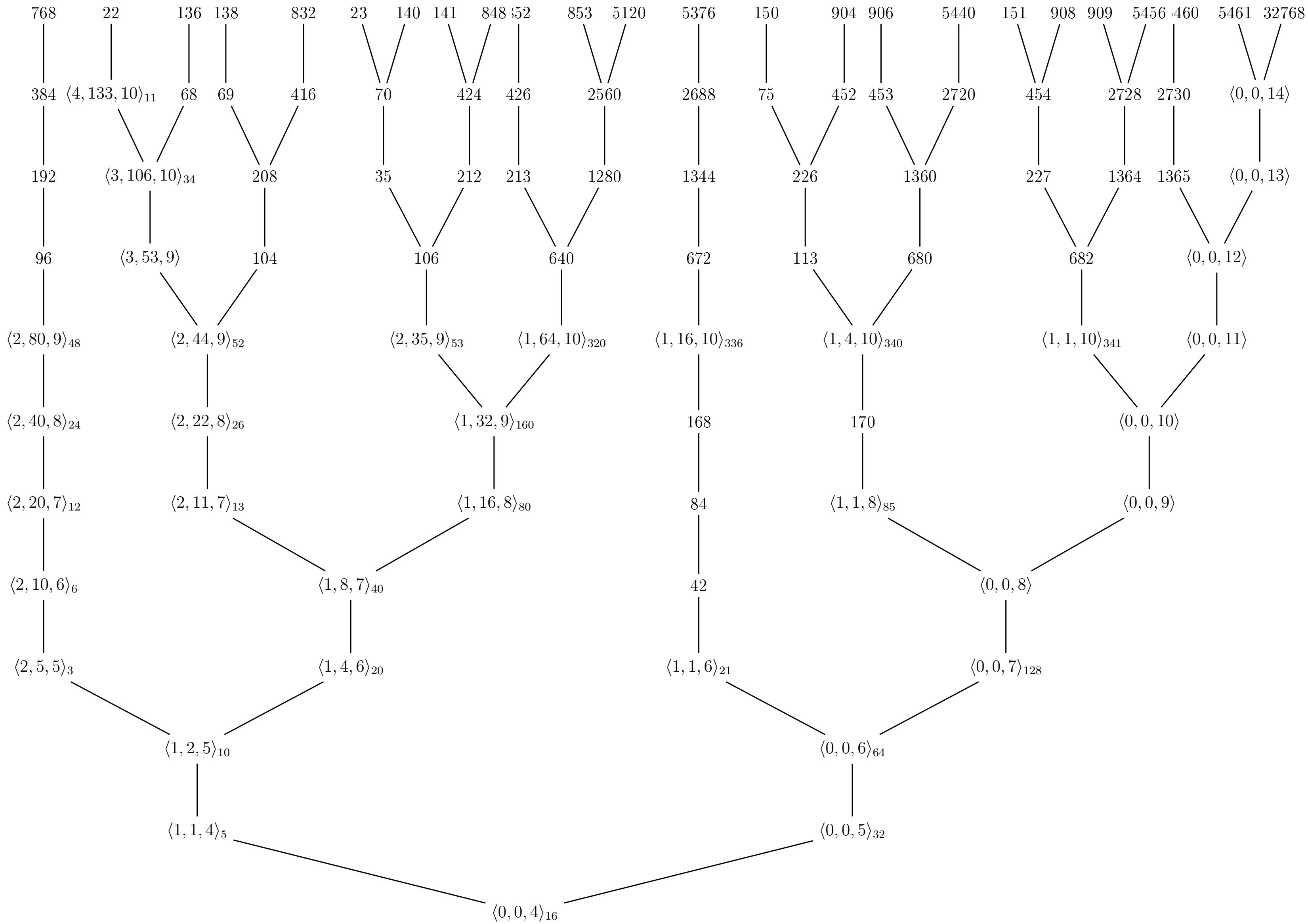


Fig. 3. Tree of triples (levels 4 – 15)

Subsequent algorithm $Gr3$ exposes the history of the calculations of x, y, z.

```
      var n,i,aux :integer ; k,m,X,Y,Z : arrayof integer;
      Γ3: | i := 0; k_i := exp(n,2); m_i := n/2^{k_i}; Z_i := k_0; Y_i, X_i := 0; |
Gr3:  while n·3^i + Y_i ≠ 2^{Z_i} do
          | aux:=3*m_i+1;
          | i := i + 1 ;
      Δ3: | k_i := exp(aux,2); m_i :=aux/2^{k_i};
          | Y_i := 3Y_{i-1} + 2^{Z_{i-1}}; Z_i := Z_{i-1} + k_i; X_i := i;
      od
```

See some properties of the algorithm $Gr3$.

Lemma 10 *Both algorithms $Gr2$ and $Gr3$ are equivalent with respect to the halting property.*
For every element n after each i-th iteration of algoritm $Gr3$, the following formulas are satisfied

$\varphi: \ n \cdot 3^i + Y_i = m_i \cdot 2^{Z_i}$	$X_i = i$
$Z_i = \sum_{j=0}^{i} k_j$	$Y_i = \sum_{j=0}^{i-1} \left(3^{i-1-j} \cdot 2^{Z_j} \right)$

where the sequences $\{m_i\}$and $\{k_i\}$ are determined by the recurrence (rec2).
in other words, the following formula is valid in the structure $\mathfrak{N}$

$$\mathfrak{N} \models \Gamma_3 \bigcap \{\textbf{if } m_i \neq 1 \textbf{ then } \Delta_3 \textbf{ fi}\} \, \varphi$$

Remark 11 *Hence, for every element n algorithm $Gr3$ calculates an increasing, monotone sequence of triples $\langle i \, (= X_i), Y_i, Z_i \rangle$.*

Remark 12 *We can say informally that the algorithm $Gr3$ performs as follow*

$i := 0$;
while $n \notin W_i$ ***do*** $i := i + 1$ ***od***

Note, $\{Gr_3\} \, (n \in W_i)$

Hotel contains rooms of any natural number. Let $n = 2^i \cdot (2j+1)$. It means that the room number n is located in tower number j on the floor number i . Each tower is equipped with an elevator (shown as a green line). Moreover, each tower is connected to another by a staircaise that connects numbers $k = 2j+1$ and $3k+1$. This is shown as a red arrow $\overrightarrow{\langle k, 3k+1 \rangle}$.

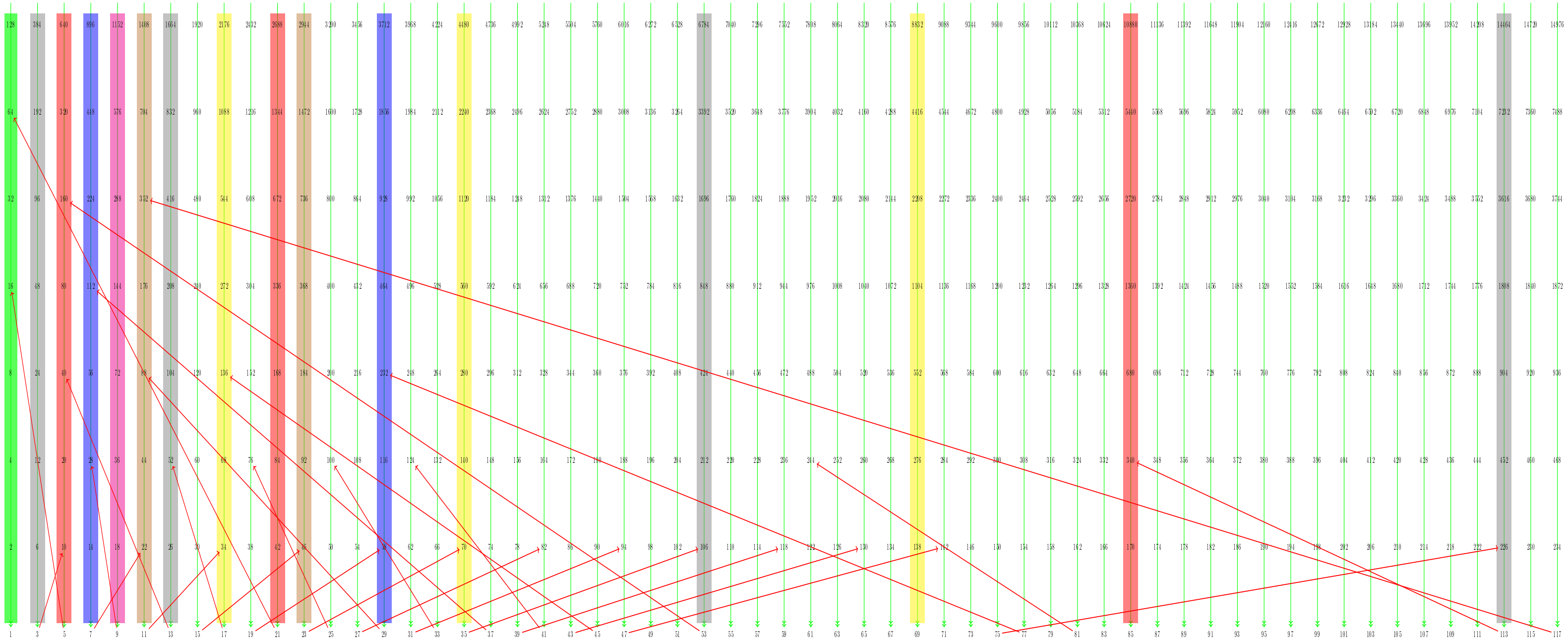


Fig. 4. Hotel Collatz

Definition 13 (Hotel Collatz) *The graph $HC = \langle V, E \rangle$ is defined as follows*

$V = N$ *i.e. the set of vertices is the set of standard, reachable, natural numbers*

$E = \{\overrightarrow{\langle k, p \rangle} : \exists_p\, k = p + p\} \cup \{\overrightarrow{\langle k, 3k+1 \rangle} : \exists_p k = p + p + 1\}$ *are edges of the graph*

> **Note.** Don't forget, our drawing is only a small fragment of the infinite HC structure. The picture shows a small part of red arrows. We drew only those red arrows that fit entirely on a page.

Conjecture 14 *The hotel Collatz is an infinite, connected, acyclic graph, i.e. it is a tree. Number 1 is the root of the tree.*

Making use of the definition 6 one can formulate the following

Conjecture 15 *The set $\overline{W} \stackrel{df}{=} \bigcup_{x \in N} W_x$ is a partition of the set N of nodes of Hotel Collatz.*

4 On finite and infinite computations of Collatz algorithm

QUESTION Can some computation contain both reachable and unreachable elements?
No. The subset of reachable elements is closed with respect of division by 2 and multiplication by 3. The same observation applies to the set of unreachable elements. We know, cf. subsection 6.1 that computations of nonreachable elements are infinite.

4.1 Finite computations

Let $\mathfrak{M} = \langle M; 0, 1, +, = \rangle$ be any algebraic structure that is a model of elementary theory of addition of natural numbers, c.f. subsection 6.2.
Denotation. Let $\theta(x, y)$ be a formula. The expression $(\mu x)\theta(x, y)$ denotes the least element $x(y)$ such that the value of the formula is truth.

EXAMPLE. $(\mu x)(x + x > y)$.

The following lemma gathers the facts established earlier.

Lemma 16 *Let n be an arbitrary element of the structure $\mathfrak{M}$. The following conditions are equivalent*

(i) *The sequence $n_0 = n$ and $n_{i+1} = \begin{cases} n_i \div 2 & \textit{when} \;\; n_i \mod 2 = 0 \\ 3n_i + 1 & \textit{when} \;\; n_i \mod 2 = 1 \end{cases}$ determined by the recurrence (rec1) contains an element $n_j = 1$*

(ii) *The computation of the algorithm Cl is finite.*

(iii) *The sequence $m_0 = \frac{n}{2^{k_0}}$ and $m_{i+1} = \frac{3m_i+1}{2^{k_i}}$ determined by the recurrence (rec2) stabilizes, i.e. there exist l such that $m_k = 1$ for all $k > l$*

(iv) *The computation of the algorithm Gr is finite.*

(v) *The computation of the algorithm $Gr1$ is finite and the subsequent values of the variables M_i and K_i satisfy the recurrence (rec2) .*

(vi) *The computation of the algorithm $Gr2$ is finite and the subsequent values of the variables m_i and k_i satisfy the recurrence (rec2). The formula $n \cdot 3^x + y = m_i \cdot 2^z$ holds after each iteration of **while** instruction, i.e. it is the invariant of the program Δ_2. The final valuation of variables x, y, z and n satisfies the equation $n \cdot 3^x + y = 2^z$.*

(vii) *The computation of the algorithm $Gr3$ is finite.*
The subsequent values of the variables m_i and k_i satisfy the recurrence (rec2) .
The subsequent values of the variables X_i, Y_i, Z_i form a monotone, increasing sequence of triples.
The formula $n \cdot 3^{X_i} + Y_i = m_i \cdot 2^{Z_i}$ is satisfied after each i-th iteration of the program $Gr3$, i.e. the value of the following exression $\{\Gamma_3;\ \Delta_3^i\}(X_i + Z_i)$ is the total number of operations excuted. The value of the variable Y_i encodes the history of the computation till the i-th iteration of Δ_3

Suppose that for a given element n the computationof algorithm $Gr2$ is finite.

Let $\bar{x} = (\mu x)\left(n \cdot 3^x + \left[\sum_{j=0}^{x-1}\left(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)\right] = 2^{\sum_{j=0}^{x} k_j}\right)$. Put $\bar{y} = \sum_{j=0}^{\bar{x}-1}\left(3^{\bar{x}-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)$ and $\bar{z} = \sum_{j=0}^{\bar{x}} k_j$.
We present the algorithm IC', which is a slightly modified version of the algorithm IC devised in [MS21] .

$$IC' : \left\{\begin{array}{l} \textbf{var}\ \mathrm{x}, \mathrm{y}, \mathrm{z}, k : integer, Err : Boolean; \\ Err := false; \\ \textbf{while}\ \mathrm{x}+\mathrm{y}+\mathrm{z} \neq 0\ \ \textbf{do} \\ \quad Tr : \boxed{\begin{array}{l} \textbf{if}\ \big(odd(\mathrm{y}) \wedge ((\mathrm{x}=0)\ \vee\ (\mathrm{y} < 3^{\mathrm{x}-1}))\big) \\ \quad \textbf{then}\ Err := true;\ \textbf{exit} \\ \textbf{fi}; \\ \ \mathrm{x} := \mathrm{x}-1;\ \mathrm{y} := \mathrm{y} - 3^{\mathrm{x}};\ k := expo(\mathrm{y}, 2); \\ \mathrm{y} := \frac{\mathrm{y}}{2^k};\ \mathrm{z} := \mathrm{z} - k; \end{array}} \\ \textbf{od} \end{array}\right\} \qquad \text{(IC')}$$

We observe the following fact

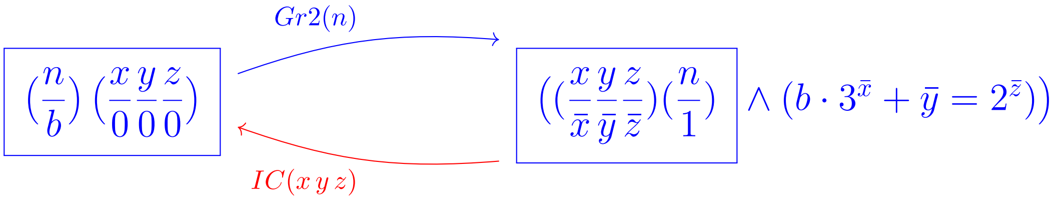
Lemma 17 *For every element* n

$$(n=b) \wedge \{Gr2\}\Big(\big((x=\bar{x} \wedge y=\bar{y} \wedge z=\bar{z}) \wedge (b \cdot 3^{\bar{x}} + \bar{y} = 2^{\bar{z}})\big) \Rightarrow \{IC'\}(x=y=z=0)\Big)$$

and

$$\begin{array}{l} (x=\bar{x} \wedge y=\bar{y} \wedge z=\bar{z}) \wedge (b \cdot 3^{\bar{x}} + \bar{y} = 2^{\bar{z}})\big) \wedge \\ \Big(\{IC'\}(x=y=z=0) \implies (n=b) \implies \{Gr2\}(x=\bar{x} \wedge y=\bar{y} \wedge z=\bar{z})\Big) \end{array}$$

The contents of this lemma are best explained by the commutativity of the diagram below.



Proof. The proof makes use of two facts:

1) $\quad even(n) \equiv even(\sum_{j=0}^{x-1}(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}))$

2) $\sum_{j=0}^{x-1}(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}) = 2^{k_0} \cdot (3^{x-1} + 2^{k_1} \cdot (3^{x-2} + 2^{k_2} \cdot (\cdots + 2^{k_x} \cdot 3^0)))$

One can prove this lemma by induction w.r.t. number of encountered odd numbers.
The thesis of the lemma is very intuitive. Look at the Collatz hotel Fig. 4. The lemma states that for every room number n the two conditions are equivalent (1) there is a path from room number n to the exit, which is located near room number 1, (2) there is a path from entry to the hotel near room number 1 to the room number n. It is clear that such a path must be complete, no jumps are allowed. ■

Lemma 18 *For every element n the following conditions are equivalent*

(i) *computation of Collatz algorithm Cl is finite,*
(ii) *there exists the* LEAST *element x such that the following equality holds*

$$n \cdot 3^x + \left(\sum_{j=0}^{x-1} 3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l} \right) = 2^{\sum_{j=0}^{x} k_j}. \qquad \text{(Mx)}$$

where the sequence $\{k_i\}$ is determined by the element n in accordance to the recurrence (rec2).

Proof. The implication $(i) \Rightarrow (ii)$ follows from lemma 16(vii).
Consider the inverse implication $(ii) \Rightarrow (i)$. If (ii) holds, then the computation of program Gr_3 reaches 1 after x-th iteration of internal instruction Δ_3. The value of m_x is then 1. The computation performs x nultiplications and $z = \sum_{j=0}^{x} k_j$ divisions.

The value of $y = \left(\sum_{j=0}^{x-1} (3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}) \right)$ codes the history of the computation. ■

Which means: x_0 is the number of multiplication by 3, $z = \sum_{j=0}^{x} k_j$ is the total number of divisions by 2 and for every $0 \leq j \leq x-1$ the number k_j is the number of divisions by 2 excuted in between the j-th and $j+1$-th execution of multiplication by 3.

The algorithm Cl executes $x + z$ iterations.

Remark 19 *Let n be a reachable natural number. The least triple $\langle x, y, z \rangle$ such that $n \cdot 3^x + y = 2^z$ is greater than $\langle 0, 2^{\lceil \log n \rceil + 1} - n, \lceil \log n \rceil + 1 \rangle$.*

The lemma 18 gives the halting formula i.e. a satisfactory and necessary condition for the computation of the Collatz program to be finite.
We shall summarize the considerations on finite computations in the following commutative diagram.

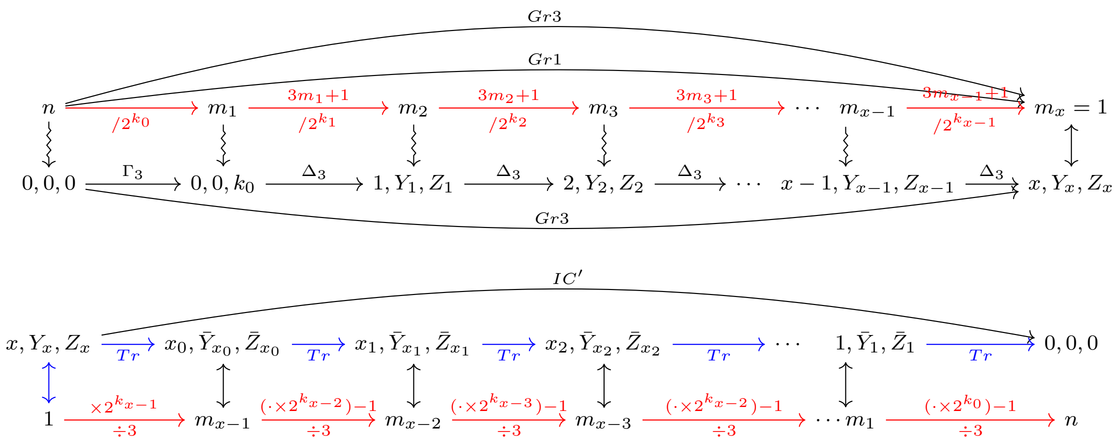


Fig. 5. Case of finite computation illustrated

Upper row, (with red arrows) represents computation of $Gr1$,
elements k_i and m_i are calculated in accordance with the recurrence (rec2)
rows 1 and 2 show computation of $Gr3$, the subsequent triples are $X_{i+1} = i+1, Y_{i+1} = 3Y_i + 2_i^Z, Z_{i+1} = Z_i + k_i$
third row (blue arrows) shows computation of algorithm IC on triples, $\bar{Y}_x = Y_x$ and $\bar{Z}_x = Z_x$ and for $i = x, \ldots, 1$ we have $\bar{Z}_{i-1} = \bar{Z}_i - k_i$ and $\bar{Y}_{i-1} = (\bar{Y}_i / 2^{k_i}) - 3^{i-1}$

Do infinite computations exist?
There are two answers *yes* and *no*.

Yes Imagine your computer is (*maliciously*) handled by a hacker. It can be done by preparing its hardware or software (e.g. someone modifies the class Int in Java). To hide the damage from the user, the hacker may come with a correct proof that all axioms of natural numbers (e.g. of Presburger's system) are valid. Yet, an execution of the Collatz algorithm for some n, will not necessarily terminate. See subsections 4.4 and 6.1.

No If argument of Collatz procedure is a standard (i.e. reachable) natural number then the computation is finite.

We shall use the properties of strata, see definition 6, and remark 7. A look on figures 2 and 4 may help.

Lemma 20 *If $n \neq 0$ is a standard, reachable natural number then the Collatz computation starting with n is finite.*

Proof. A) (*There is no cycle.*)
Suppose that for a certain element n the Collatz computation contains a cycle $\vec{b} = (b_1, b_2, \ldots, b_q = b_1)$. Consider the subsequence $\vec{a} = (a_1, \ldots, a_k)$ of $\vec{b}$ that results by throwing off all even numbers b_i. For every $1 \leq i < k$ we have $a_{i+1} = \frac{3a_i+1}{2^{expo(3a_i+1,2)}}$. From the definition 6 of strata we have: 1° every number $n \in N$ belongs to certain set W_j and 2° if $a_i \in W_j$ then $a_{i+1} \in W_{j-1}$. 3°The strata of

$$
\begin{array}{ccccccc}
a1 & \xrightarrow[/2^{expo(a_1,2)}]{3a_1+1} & a_2 & \xrightarrow[/2^{expo(a_2,2)}]{3a_2+1} \cdots \xrightarrow[/2^{expo(a_{k-2},2)}]{3a_{k-2}+1} & a_{k-1} & \xrightarrow[/2^{expo(a_{k-1},2)}]{3a_{k-1}+1} & a_k \\
\downarrow \in & & \downarrow \in & & \downarrow \in & & \downarrow \in \\
W_x & & W_{x-1} & \ldots & W_{x-k+2} & & W_{x-k+1}
\end{array}
$$

different indices are disjoint $x \neq y \implies W_x \cap W_y = \emptyset$. Therefore for every number $n \in N$ its Collatz computation is cycle-free.

B) (*There is no unboud infinite computation.*)
Let $\widehat{W_j} \stackrel{df}{=} \bigcup_{i=0}^{j} W_i$ be the union of consecutive strata W_i, where $1 \leq i \leq j$.

Let $Mg(j) \stackrel{df}{=} (\mu x) \left\{x + 1 \notin \widehat{W_j}\right\}$ i.e. x is the biggest number in the initial segment of the set $\widehat{W_j}$ that contains all numbers $0, \ldots, x$.
It means that

$$\forall_{m \leq Mg(j})m \in \widehat{W_j}$$

Consider the sequence $\left\{\widehat{W_j}\right\}_{j \in Nat}$ of sets $\widehat{W_j}$. The sequence of numbers $Mg(j)$ is monotone, non-decreasing.

$$\forall_j Mg(j) \leq Mg(j+1)$$

Example. The table below shows a couple of values of the function Mg_j

j	0	1	2	3	4	5	6	7	8	9
$\min(W_j)$	1	5	3	17	11	7	9	61	?	?
$Mg(j)$	2	2	6	6	6	8	14	14	14	14

Obviously, for every natural number n there is number j such that $n < Mg(j)$, hence $n \in \widehat{W_j}$. This proves that for every natural number n its Collatz computation is finite. ■

We can state the same fact in another way, if there is unbound infinite computation for a number n then for every natural number j the inequality $n > Mg(j)$ holds.
This happens only if n is a non-reachable element of the structure $\mathfrak{M}$ that is a non-standard model of Presburger theory of addition of natural numbers, c.f. the subsection 4.4. **q.e.d.**

4.3 Collatz theorem

Till now we proved that Collatz conjecture is valid in the structure $\mathfrak{N}$ of standard (reachable) natural numbers.

Lemma 21 *Let n be any standard element od the structure $\mathfrak{N}$. The computation of Collatz algorithm Cl that begins with n is finite.*

Proof. The proof follows immediately from the lemmas 18 and 20. ■

Corollary 22 *Conjectures 3, 14 and 15 formulated above are valid statements.*

Now, making use of the completeness theorem of calculus of programs we argue that the terminatiom formula of the Collatz algorithm possedes a proof in the algorithmic theory of natural numbers $\mathcal{ATN}$.
Let Θ_{CL} be the following formula of $\mathcal{ATN}$

$$\underbrace{\{n := 1\} \bigcap \{n := n+1\}}_{\forall_{n \in N,\, n \geq 1}} \bigcup \left\{ \begin{array}{l} \textbf{if } n \neq 1 \textbf{ then} \\ \quad \textbf{if } odd(n) \textbf{ then} \\ \quad\quad n := 3 * n + 1 \\ \quad \textbf{else} \\ \quad\quad n := n \div 2 \\ \quad \textbf{fi} \\ \textbf{fi} \end{array} \right\} (n = 1) \qquad (\Theta_{CL})$$

The formula expresses the termination property of Collatz algorithm. The lemma 21 states that the formula Θ_{CL} is valid in the standard model $\mathfrak{N}$.

Theorem 23 *The formula Θ_{CL} is a theorem of the theory $\mathcal{ATN}$.*

Proof. Recall that the $\mathcal{ATN}$ theory is categorical. It means that every model of this theory is isomorphic to the standard model $\mathfrak{N}$.
The sentence Θ_{CL} is valid in $\mathfrak{N}$, i.e. in every model of$\mathcal{ATN}$.
By the completeness property of the calculus of programs, the theory contains the proof of the sentence Θ_{CL}. ■

4.4 A counterexample

We argue, that the formulation of the Collatz problem requires more precision. For there are several algebraic structures that can be viewed as structure of natural numbers of addition. Some of them admit infinite computations of Collatz algorithm.
We recall less known fact: arithmetic (i.e. first-order theory of natural numbers) has standard (*Archimedean*) model $\mathfrak{N}$ as well as

another *non-Archimedean* model $\mathfrak{M}$[3]. The latter structure allows for the existence of infinitely great elements.
Goedel's incompleteness theorem shows that there is no elementary theory T of natural numbers, such that every model is isomorphic to the standard model.
Two things are missing from the commonly accepted texts: 1) What do we mean by proof? 2) what properties of natural numbers can be used in the proof? We recall an algebraic structure $\mathfrak{M}$ that models [Grz71] all axioms of elementary theory of addition of natural numbers, yet it admits unreachable elements [Tar34]. It means that the model contains element ε, such that the computation of Collatz algorithm that starts with ε is infinite.

Example of a finite execution

$$\langle 13,0\rangle \overset{\times 3+1}{\to} \langle 40,0\rangle \overset{\div 2}{\to} \langle 20,0\rangle \overset{\div 2}{\to} \langle 10,0\rangle \overset{\div 2}{\to}, \langle 5,0\rangle \overset{\times 3+1}{\to} \langle 16,0\rangle \overset{\div 2}{\to} \langle 8,0\rangle \overset{\div 2}{\to} \langle 4,0\rangle \overset{\div 2}{\to} \langle 2,0\rangle \overset{\div 2}{\to} \langle 1,0\rangle$$

Example of an infinite execution

$$\langle 8,\frac{1}{2}\rangle \overset{\div 2}{\to} \langle 4,\frac{1}{4}\rangle \overset{\div 2}{\to} \langle 2,\frac{1}{8}\rangle \overset{\div 2}{\to} \langle 1,\frac{1}{16}\rangle \overset{\times 3+1}{\to} \langle 4,\frac{3}{16}\rangle \overset{\div 2}{\to} \langle 2,\frac{3}{32}\rangle \overset{\div 2}{\to} \langle 1,\frac{3}{64}\rangle, \overset{\times 3+1}{\to} \langle 4,\frac{9}{64}\rangle \overset{\div 2}{\to} \langle 2,\frac{9}{128}\rangle \overset{\div 2}{\to} \cdots$$

As you can guess, the data structure contains pairs $\langle k, w\rangle$ where k is an integer and w is a non-negative, rational number. The addition operation is defined componentwise. A pair $\langle k, w\rangle$ divided by 2 returns $\langle k \div 2, w \div 2\rangle$.
The reader may prefer to think of complex numbers instead of pairs, e.g. $(2 + \frac{9}{128}i)$ may replace the pair $\langle 2, \frac{9}{128}\rangle$.
The following observation seems to be of importance:.

Remark 24 *There exists an infinite computation* **c** *of Collatz algorithm in the structure* $\mathfrak{M}$, *such that the computation* **c** *does not contain a cycle, and the sequence of pairs is not diverging into still growing pairs. The latter means, that there exist two numbers* $l_1 \in \mathcal{Z}$ *and* $l_2 \in \mathcal{Q}$, *such that for every step* $\langle k, v\rangle$ *of computation* **c**, *the inequalities hold* $k < l_1 \wedge v < l_2$.

More details can be found in subsection 6.1.

---

[3] A. Tarski [Tar34] confirms that S. Jaśkowski observed (in 1929) that the subset of complex numbers $M \overset{df}{=} \{a + bi \in \mathbb{C} \colon (a \in \mathbb{Z} \wedge b \in \mathbb{Q} \wedge (b \geq 0 \wedge (b = 0 \implies a \geq 0))\}$ satisfies all axioms of Presburger arithmetic.

5 Final remarks

The contribution presented here leaves some open questions: first of all the cost of the algorithm Cl remains to be estimated. The lower bound is obviously $O(x+z)$. A tight upper bound remains to be found .

Another goal, that will take more time, is to write a complete syntactical (i.e. free of any semantical considerations, like studies of computation) proof of Collatz theorem. We expect that the proof will pass the checking by a proof-checker proper for calculus of programs $\mathcal{AL}$ [4]. The subsections 6.5 and 6.6 contribute to this work.

5.1 Historical remarks

Pàl Erdős said on Collatz conjecture: *"Mathematics may not be ready for such problems."*
We disagree. In our opinion a consortium of Alfred Tarski, Kurt Goedel and Stephen C. Kleene was able to solve the Collatz conjecture in 1937.

- Mojżesz Presburger has proved the completeness and decidability of arithmetic of addition of natural numbers in 1929.
- In the same year Stanisław Jaśkowski found a non-standard model of Presburger theory (see a note of A. Tarski of 1934).
- Kurt Gödel (1931) published his theorem on incompleteness of Peano's theory. His result is of logic, not an arithmetic fact.
- Thoralf Skolem (in 1934) wrote a paper on the non-characterization of the series of numbers by means of a finite or countably infinite number of statements with exclusively individual variables [Sko34]
  Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich

[4] such proof-checker does not exists yet and it is to be built

oder abzählbar unendlich vieler Aussagen mit ausschließlich Zahlenvariablen , Fundamenta Mathematicae, ,**23**,1, 150–161, `http://matwbn.icm.edu.pl/ksiazki/fm/fm23/fm23115.pdf`

- Stephen C. Kleene has shown (in 1936) that any recurrence that defines a computable function can be replaced by the operation of effective minimum (nowadays one can say every recursive function in the integers, is programmable by means of **while** instruction).
- Summing up, it seems that P. Erdős overlooked the computability theory, his colleagues - professors Rozsa Peter and Laszlo Kalmar (specialists in the theory of recursive functions) were able to point it out to him.

Andrzej Mostowski had a hope that many arithmetic theorems independent of the Peano axioms should be found. Collatz theorem is an example. The theorem on termination of Euclid's algorithm is another example of a theorem which is valid and unprovable in Peano theory. The law of Archimedes is yet another example. Note, both theorems need to be stated as algorithmic formulas, there is no first-order formula that expresess the termination property of Euclid's algorithm or law of Archimedes.

Acknowlegments

6 Suplements

For the reader's convenience, in this section we have included some definitions, some useful theorems, and samples of proofs in algorithmic natural number theory.

6.1 A structure with counterexamples

where Collatz computations may be of infinite length

Here we present some facts that are less known to the IT community.
These facts may seem strange. The reader may doubt the importance of those facts. Yet, it is worth considering, non-standard data structures do exist, and this fact has ramifications. Strange as they seem, still it is worthwhile to be aware of their existence.
Now, we will expose the algebraic structure $\mathfrak{M}$, which is a model of the theory Ar, i.e. all axioms of theory Ar are true in the structure $\mathfrak{M}$. First we will describe this structure as mathematicians do, then we will write a class (i.e. a program module) implementing this structure.

Mathematical description of the structure

$\mathfrak{M}$ is an algebraic structure

$$\mathfrak{M} = \langle M; \underline{\mathbf{0}}, \underline{\mathbf{1}}, \oplus; =\rangle \qquad \text{(NonStandard)}$$

such that M is a set of complex numbers $k + \imath w$, i.e. of pairs $\langle k, w\rangle$, where element $k \in \mathbb{Z}$ is an integer, and element $w \in \mathbb{Q}^+$ is a rational, non-negative number $w \geq 0$ and the following requirements are satisfied:

(i) for each element $k + \imath w$ if $w = 0$ then $k \geq 0$,

(ii) $\underline{\mathbf{0}} \stackrel{df}{=} \langle 0 + \imath 0\rangle$,

(iii) $\underline{\mathbf{1}} \stackrel{df}{=} \langle 1. + \imath 0\rangle$,

(iv) the operation $\oplus$ of addition is determined as usual

$$(k + \imath w) \oplus (k' + \imath w') \stackrel{df}{=} (k + k') + \imath(w + w').$$

(v) the predicate $=$ denotes as usual identity relation.

Lemma 25 *The algebraic structure $\mathfrak{M}$ is a model of first-order arithmetic of addition of natural numbers $\mathcal{T}$, cf. next subsection 6.2*

The reader may check that every axiom of the $\mathcal{T}$ theory (see definition32, p.33), is a sentence true in the structure $\mathfrak{M}$.

The substructure $\mathfrak{N} \subset \mathfrak{M}$ composed of only those elements for which $w = 0$ is also a model of the theory $\mathcal{T}$.
It is easy to remark that elements of the form $\langle k, 0\rangle$ may be identified with natural numbers k, $k \in N$. Have a look at table 1

The elements of the structure $\mathfrak{N}$ are called *reachable*, for they enjoy the following algorithmic property

$$\forall_{n\in N}\,\{y := \mathbf{0}; \mathbf{while}\ y \neq n\ \mathbf{do}\ y := y + \mathbf{1}\ \mathbf{od}\}(y = n)$$

The structure $\mathfrak{M}$ is not a model of the $\mathcal{ATN}$, algorithmic theory of natural numbers, cf . subsection 6.4. .Elements of the structure $\langle k, w\rangle$. such as $w \neq \mathbf{0}$ are *unreachable*. i.e. for each element $x_0 = \langle k, w\rangle$ such that $w \neq 0$ the following condition holds

$$\neg\{y := \mathbf{0}; \mathbf{while}\ y \neq x_0\ \mathbf{do}\ y := y + \mathbf{1}\ \mathbf{od}\}(y = x_0)$$

The subset $\mathfrak{N} \subset \mathfrak{M}$ composed of only those elements for which $w = 0$ is a model of the theory $\mathcal{ATN}$ c.f. subsection 6.4. The elements of the structure $\mathfrak{N}$ are called *reachable*. A very important theorem of the foundations of mathematics is

Fact 1 *The structures $\mathfrak{N}$ and $\mathfrak{M}$ are not isomorphic. See [Grz71], p. 256.*

As we will see in a moment, this fact is also important for IT specialists.

An attempt to visualize structure $\mathfrak{M}$ is presented in the form of table 1. The universe of the structure $\mathfrak{M}$ decomposes onto two disjoint subsets (one green and one red). Every element of the form $\langle k, 0\rangle$ (in this case $k > 0$) represents the natural number k. Such elements are called *reachable* ones. Note,

Definition 26 *An element n is a standard natural number (i.e. is* *reachable) iff the program of adding ones to initial zero terminates*

$$n \in N \overset{df}{\Leftrightarrow} \{q := \mathbf{\mathit{0}};\ \mathbf{\mathit{while}}\ q \neq n\ \mathbf{\mathit{do}}\ q := q + \mathbf{\mathit{1}}\ \mathbf{\mathit{od}}\}(q = n)$$

or, equivalently

$$n \in N \overset{df}{\Leftrightarrow} \{q := \mathbf{\mathit{0}}\} \bigcup \{\mathbf{if}\ n \neq q\ \mathbf{then}\ q := q + \mathbf{\mathit{1}}\ \mathbf{fi}\}(q = n)$$

Table 1
Model $\mathfrak{M}$ of Presburger arithmetic consists of complex numbers $a + \imath b$ where $b \in Q^+$ and $a \in Z$, additional condition: $b = 0 \Rightarrow a \geq 0$. Definition of order $n > m \overset{df}{\equiv} \exists_{u \neq 0}\, m + u = n$. Invention of S. Jaśkowski (1929).

STANDARD (*reachable*) elements	*Unreachable* (INFINITE) elements							
					$\cdots$			
	$-\infty\cdots$	$-11+\imath 2$	$-10+\imath 2$	$\cdots$	$0+\imath 2$	$1+\imath 2$	$2+\imath 2$	$\cdots\infty$
					$\cdots$			
	$-\infty\cdots$	$-11+\imath\frac{53}{47}$	$-10+\imath\frac{53}{47}$	$\cdots$	$0+\imath\frac{53}{47}$	$1+\imath\frac{53}{47}$	$2+\imath\frac{53}{47}$	$\cdots\infty$
					$\cdots$			
	$-\infty\cdots$	$-11+\imath\frac{28}{49}$	$-10+\imath\frac{28}{49}$	$\cdots$	$0+\imath\frac{28}{49}$	$1+\imath\frac{28}{49}$	$2+\imath\frac{28}{49}$	$\cdots\infty$
					$\cdots$			
	$-\infty\cdots$	$-11+\imath\frac{3}{47}$	$-10+\imath\frac{3}{47}$	$\cdots$	$0+\imath\frac{3}{47}$	$1+\imath\frac{3}{47}$	$2+\imath\frac{3}{47}$	$\cdots\infty$
					$\cdots$			
$0\ 1\ 2\ \cdots\ 101\ \cdots\ \infty$								

Note that the subset that consists of all non-reachable elements is well separated from the subset of reachable elements. Namely, every reachable natural number is less that any unreachable one. Moreover, there is no least element in the set of unreachable elements. I.e. the principle of minimum does not hold in the structure $\mathfrak{M}$.
Moreover, for every element n its computation contains either only standard, reachable numbers or is composed of only unreachable elements. This remark will be of use in our proof.

Remark 27 *For every element n the whole Collatz computation is either in green or in reed quadrant of the table 1.*

Elements of the structure $\mathfrak{M}$ are ordered as usual

$$\forall_{x,y}\ x < y \stackrel{df}{=} \exists_{z\neq\mathbf{0}}\ x + z = y.$$

Therefore, each reachable element is smaller than every unreachable element.
The order defined in this way is the lexical order. (Given two elements p and q, the element lying higher is bigger, if both are of the same height then the element lying on the right is bigger.)
The order type is $\omega + (\omega^* + \omega) \cdot \eta$

Remark 28 *The subset of unreachable elements (red ones on the table 1) does not obey the principle of minimum.*

Definition in programming language

Perhaps you have already noticed that the $\mathfrak{M}$ is a computable structure. The following is a class that implements the structure $\mathfrak{M}$. The implementation uses the integer type, we do not introduce rational numbers explicitly.

```
unit StrukturaM: class;
  unit Elm: class(k,li,mia: integer);
  begin
    if mia=0 then raise Error fi;
    if li * mia <0 then raise Error fi;
    if li=0 and k<0 then raise Error fi;
  end Elm;
  add: function(x,y:Elm): Elm;
  begin
    result := new Elm(x.k+y.k, x.li*y.mia+x.mia*y.li, x.mia*y.mia )
  end add;
  unit one : function:Elm;  begin  result:= new Elm(1,0,2)  end one;
  unit zero : function:Elm;  begin  result:= new Elm(0,0,2)  end zero;
  unit eq: function(x,y:Elm): Boolean;
  begin
    result := (x.k=y.k) and (x.li*y.mia=x.mia*y.li )
  end eq;
end StrukturaM
```

The following lemma expresses the correctness of the implementation with respect to the axioms of Presburger arithmetic $\mathcal{AP}$ (c.f. subsection 6.2) treated as a specification of a class (i.e. a module of program).

Lemma 29 *The structure $\mathfrak{E} = \langle E, add, zero, one, eq\rangle$ composed of the set $E = \{o\ object : o\ inElm\}$ of objects of class Elm with the add operation is a model of the $\mathcal{AP}$ theory,*

$$\mathfrak{E} \models \mathcal{AP}$$

Infinite Collatz algorithm computation

How to execute the Collatz algorithm in StructuraM? It's easy.

```
pref StrukturaM block
   var n: Elm;
   unit odd: function(x:Elm): Boolean; ... result:=(x.k mod 2)=1 ... end odd;
   unit div2: function(x:Elm): Elm; ...
   unit 3xp1: function(n: Elm): Elm; . . . result:=add(n,add(n,add(n,one))); . . . end 3xp1;
begin
   n:= new Elm(8,1,2);
       ┌──────────────────────────────────┐
       │ while  not eq(n,one)  do         │
       │    if  odd(n) then               │
   Cl: │       n:=3xp1(n)  else n:= div2(n) │   (* a version of algorithm Cl that uses class Elm *)
       │    fi                            │
       │ od                               │
       └──────────────────────────────────┘
end block;
```

Below we present the computation of Collatz algorithm for $n = \langle 8, \frac{1}{2}\rangle$.

$$\langle 8, \frac{1}{2}\rangle,\ \langle 4, \frac{1}{4}\rangle, \langle 2, \frac{1}{8}\rangle,\ \langle 1, \frac{1}{16}\rangle,\ \langle 4, \frac{3}{16}\rangle,\ \langle 2, \frac{3}{32}\rangle,\ \langle 1, \frac{3}{64}\rangle,\ \langle 4, \frac{9}{64}\rangle,\ \langle 2, \frac{9}{128}\rangle, \cdots$$

Note, the computation of algorithm Gr for the same argument, looks simpler

$$\langle 8, \frac{1}{2}\rangle,\ \langle 4, \frac{1}{4}\rangle, \langle 2, \frac{1}{8}\rangle,\ \langle 1, \frac{1}{16}\rangle,\ \langle 1, \frac{3}{64}\rangle,\ \langle 1, \frac{9}{256}\rangle, \cdots$$

None of the elements of the above sequence is a standard natural number. Each of them is unreachable. It is worth looking at an example of another calculation. Will something change when we assign n a different object? e.g. n: = new Elm (19,2,10)?

$$\langle 19, \tfrac{10}{2}\rangle,\ \langle 58, \tfrac{30}{2}\rangle, \langle 29, \tfrac{30}{4}\rangle,\ \langle 88, \tfrac{90}{4}\rangle,\ \langle 44, \tfrac{90}{8}\rangle,\ \langle 22, \tfrac{90}{16}\rangle,\ \langle 11, \tfrac{90}{32}\rangle,\ \langle 34, \tfrac{270}{32}\rangle,\ \langle 17, \tfrac{270}{64}\rangle,$$
$$\langle 52, \tfrac{810}{64}\rangle, \langle 26, \tfrac{405}{64}\rangle, \langle 13, \tfrac{405}{128}\rangle, \langle 40, \tfrac{1215}{128}\rangle, \langle 20, \tfrac{1215}{256}\rangle, \langle 10, \tfrac{1215}{256}\rangle, \langle 5, \tfrac{1215}{512}\rangle, \langle 16, \tfrac{3645}{512}\rangle, \langle 8, \tfrac{3645}{1024}\rangle,$$
$$\langle 4, \tfrac{3645}{2048}\rangle, \langle 2, \tfrac{3645}{4096}\rangle, \langle 1, \tfrac{3645}{8192}\rangle, \langle 4, \tfrac{3*3645}{8192}\rangle, \langle 2, \tfrac{3645*3}{2*8192}\rangle, \langle 1, \tfrac{3*3645}{4*8192}\rangle, \langle 4, \tfrac{9*3645}{4*8192}\rangle, \cdots$$

And one more computation.

$\langle 19,0\rangle$, $\langle 58,0\rangle$, $\langle 29,0\rangle$, $\langle 88,0\rangle$, $\langle 44,0\rangle$, $\langle 22,0\rangle$, $\langle 11,0\rangle$, $\langle 34,0\rangle$, $\langle 17,0\rangle$, $\langle 52,0\rangle$, $\langle 26,0\rangle$, $\langle 13,0\rangle$, $\langle 40,0\rangle$, $\langle 20,0\rangle$, $\langle 10,0\rangle$, $\langle 5,0\rangle$, $\langle 16,0\rangle$, $\langle 8,0\rangle$, $\langle 4,0\rangle$, $\langle 2,0\rangle$, $\langle 1,0\rangle$.

Corollary 30 *The structure $\mathfrak{M}$, which we have described in two different ways, is the model of the $\mathcal{AP}$ theory with the non-obvious presence of unreachable elements in it.*

Corollary 31 *The halting property of the Collatz algorithm cannot be proved from the axioms of the $\mathcal{T}$ theory, nor from the axioms of $\mathcal{AP}$ theory.*

6.2 Presburger arithmetic

Presburger arithmetic is another name of elementary theory of natural numbers with addition.
We shall consider the following theory , cf. [Pre29],[Grz71] p. 239 and following ones.

Definition 32 *Theory $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, Ax\rangle$ is the system of three elements:*

$\mathcal{L}$ *is a language of first-order. The alphabet of this language consist of: the set V of variables, symbols of operations: $0, S, +$, symbol of equality relation $=$, symbols of logical functors and quantifiers, auxiliary symbols as brackets ...*
The set of well formed expressions is the union of te set T of terms and the set of formulas F.
The set T is the least set of expressions that contains the set V and constants 0 and 1 and closed with respect to the rules: if two expressions τ_1 and τ_2 are terms, then the expression $(\tau_1 + \tau_2)$ is a term too.
The set F of formulas is the least set of expressions that contains the equalities (i.e. the expressions of the form $(\tau_1 = \tau_2)$) and closed with respect to the following formation rules: if ex-

pressions α and β are formulas, then the aexpression of the form

$$(\alpha \vee \beta),\ (\alpha \wedge \beta),\ (\alpha \implies \beta),\ \neg\alpha$$

are also formulas, moreover, the expressions of the form

$$\forall_x \alpha,\ \exists_x \alpha$$

where x is a variable and α is a formula, are formulas too.

$\mathcal{C}$ *is the operation of consquence determined by axioms of first-order logic and the inference rules of the logic,*

Ax *is the set of formulas listed below.*

$$\forall_x\ x+1 \neq 0 \tag{a}$$
$$\forall_x \forall_y\ x+1 = y+1 \implies x = y \tag{b}$$
$$\forall_x\ x+0 = x \tag{c}$$
$$\forall_{x,y}\ (y+1)+x = (y+x)+1 \tag{d}$$
$$\Phi(0) \wedge \forall_x [\Phi(x) \implies \Phi(x+1)] \implies \forall_x \Phi(x) \tag{I}$$

The expression $\Phi(x)$ may be replaced by any formula. The result is an axiom of theory This is the induction scheme.
We augment the set of axioms adding four axioms that define a coiple of useful notions.

$$even(x) \stackrel{df}{\equiv} \exists_y\, x = y+y \tag{e}$$
$$odd(x) \stackrel{df}{\equiv} \exists_y\, x = y+y+1 \tag{o}$$
$$x\, div\, 2 = y \equiv (x = y+y\ \vee\ x = y+y+1) \tag{D2}$$
$$3x \stackrel{df}{=} x+x+x \tag{3x}$$

The theory $\mathcal{T}'$ obtained in this way is a conservative extension of theory $\mathcal{T}$.

Below we present another theory $\mathcal{AP}$ c.f. [Pre29], we shall use two facts: 1) theory $\mathcal{AP}$ is complete and hence is decidable, 2) both theories are elementarily equivalent.

Definition 33 *Theory* $\mathcal{AP} = \langle \mathcal{L}, \mathcal{C}, AxP \rangle$ *is a system of three elements :*

$\mathcal{L}$ *is a language of first-order. The alphabet of this language contains the set* V *of variables, symbols of functors :* $0, +$*, symbol of equality predicate* $=$*.*
The set of well formed-expressions is the union of set of terms T *and set of formulas* F*. The set of terms* T *is the least set of expressions that contains the set of variables* V *and the expression* 0 *and closed with respect to the following two rules: 1) if two expressions* τ_1 *and* τ_2 *are terms, then the expression* $(\tau_1 + \tau_2)$ *is also a term, 2) if the expression* τ *is a term, then the expression* $S(\tau)$ *is also a term.*

$\mathcal{C}$ *is the consequence operation determined by the axioms of predicate calculus and inference rules of first-order logic*

AxP *The set of axioms of the* $\mathcal{AP}$ *theory is listed below.*

$$\forall_x \; x + 1 \neq 0 \tag{A}$$
$$\forall_x \; x \neq 0 \implies \exists_y x = y + 1 \tag{B}$$
$$\forall_{x,y} \; x + y = y + x \tag{C}$$
$$\forall_{x,y,z} \; x + (y + z) = (x + y) + z \tag{D}$$
$$\forall_{x,y,z} \; x + z = y + z \implies x = y \tag{E}$$
$$\forall_x \; x + 0 = x \tag{F}$$
$$\forall_{x,z} \; \exists_y \; (x = y + z \vee z = y + x) \tag{G}$$
$$\forall_x \; \exists_y \; (x = y + y \vee x = y + y + 1) \tag{H2}$$
$$\forall_x \; \exists_y \; (x = y + y + y \vee x = y + y + y + 1 \vee x = y + y + y + 1 + 1) \tag{H3}$$

. . . .

$$\forall_x \exists_y \left(\begin{array}{l} x = \underbrace{y + y + \cdots + y}_{k} \vee \\ \quad x = \underbrace{y + y + \cdots + y}_{k} + 1 \vee \\ \quad\quad x = \underbrace{y + y + \cdots + y}_{k} + \underbrace{1 + 1}_{2} \vee \\ \quad\quad \ldots \\ \quad\quad\quad x = \underbrace{y + y + \cdots + y}_{k} + \underbrace{1 + 1 + \cdots + 1}_{k-2} \vee \\ \quad\quad\quad\quad x = \underbrace{y + y + \cdots + y}_{k} + \underbrace{1 + 1 + \cdots + 1}_{k-1} \end{array} \right) \qquad \text{(Hk)}$$

...

The axioms H2 -Hk ... may be given a shorter form. Let us introduce numerals, ie. the constants representing term of the form

$$\underline{\boldsymbol{2}} \stackrel{df}{=} 1 + 1$$

$$\underline{\boldsymbol{3}} \stackrel{df}{=} 1 + 1 + 1$$

...

$$\underline{\boldsymbol{k}} \stackrel{df}{=} \underbrace{1 + 1 + \ldots 1}_{k \quad times}$$

...

Now, the axioms take form

$$\forall_x \; x \, mod \, \underline{\boldsymbol{2}} = \underline{\boldsymbol{0}} \vee x \, mod \, \underline{\boldsymbol{2}} = \underline{\boldsymbol{1}} \tag{H2'}$$

$$\forall_x \; x \, mod \, \underline{\boldsymbol{3}} = \underline{\boldsymbol{0}} \vee x \, mod \, \underline{\boldsymbol{3}} = \underline{\boldsymbol{1}} \vee x \, mod \, \underline{\boldsymbol{3}} = \underline{\boldsymbol{2}} \tag{H3'}$$

$\ldots$

$$\forall_x \bigvee_{j=0}^{k-1} x \, mod \, \underline{\boldsymbol{k}} = \underline{\boldsymbol{j}} \tag{Hk'}$$

Let us recall a couple of useful theorems

F1. Theory $\mathcal{T}$ is elementarily equivalent to the theory $\mathcal{AP}$.[Pre29] [Sta84]

F2. Theory $\mathcal{AP}$ is decidable. [Pre29].

F3. The computational complexity of theory $\mathcal{AP}$, is double exponential $O(2^{2^n})$ this result belongs to Fisher and Rabin, see [MF74].

F4. Theories $\mathcal{T}$ and $\mathcal{AP}$ have non-standard model, see section 6.1, p. 27.

Now, we shall prove a couple of useful theorems of theory $\mathcal{T}$.

First, we shall show that the sentence $\forall_n \exists_{x,y,z} \, n \cdot 3^x + y = 2^z$ is a theorem of the theory $\mathcal{T}$ of addition. Operations of multiplication and power are inaccessible in the theory $\mathcal{T}$. However, we do not need them.
We enrich the theory $\mathcal{T}$ adding two functions $P2(\cdot)$ and $P3(\cdot,\cdot)$. defined in this way

Definition 34 *Two functions are defined $P2$ (of oneargument) and $P3$ (of two-arguments).*

$P2(0) \stackrel{df}{=} 1$	$P3(y,0) \stackrel{df}{=} y$
$P2(x+1) \stackrel{df}{=} P2(x) + P2(x)$	$P3(y,x+1) \stackrel{df}{=} P3(y,x) + P3(y,x) + P3(y,x)$

Lemma 35 *The definitions given above are correct, i.e. the following sentences aretheorems of the theory with two definitions*

$$\mathcal{T} \vdash \forall_x \exists_y \, P2(x) = y \quad and$$
$$\mathcal{T} \vdash \forall_{x,y,z} P2(x) = y \wedge P2(x) = z \implies y = z.$$
Similarly, the sentences $\forall_{y,x}\exists_z \, P3(y,x) = z$ *and* $\forall_{y,x,z,u} P3(y,x) = z \wedge P3(y,x) = u \implies z = u$ *are theorems of theory* $\mathcal{T}$ *.*

An easy proof goes by induction with respect to the value of variable x.

In the proof of the lemma 36 , below, we shall use the definition of the order relation

$$a < b \stackrel{df}{=} \exists_{c \neq 0} \; a + c = b.$$

Making use of the definition of function $P2$ and $P3$ we shall write the formula $P3(n,x)+y = P2(z)$ as it exppresses the same content as expression $n \cdot 3^x + y = 2^z$.

Lemma 36 *The following sentence is a theorem of the theory* $\mathcal{T}$ *enriched by the definitions of* $P2$ *and* $P3$ *functions.*

$$\forall_n \exists_{x,y,z} P3(n,x) + y = P2(z)$$

Proof. We begin proving by induction that $\mathcal{T} \vdash \forall_n \, n < 2^n$. It is easy to see that $\mathcal{T} \vdash 0 < P2(0)$. We shall prove that $\mathcal{T} \vdash \forall_n (n < P2(n) \implies (n+1 < P2(n+1))$. Inequality $n+1 < P2(n+1)$ follows from the two following inequalities $\mathcal{T} \vdash n < P2(n)$ and $\mathcal{T} \vdash 1 < P2(n)$. Hence the formula $n+1 < P2(n) + P2(n))$ is a theorem of theory $\mathcal{T}$. By definition $P2(n) + P2(n) = P2(n+1)$.
In the similar manner, we can prove the formula $\mathcal{T} \vdash \forall_n \forall_x \, P3(n,x) < P2(n + x + x)$
As a consequence we have $\mathcal{T} \vdash \forall_n \exists_{x,y,z} \; P3(n,x) + y = P2(z)$. ■

Lemma 37 *Let* $\mathfrak{M}$ *be any model of Presburger arithmetic. If there exists a triple* $\langle x, y, z \rangle$ *of reachable elements such that it satisfies the equation* $P3(n,x) + y = P2(z)$ *i.e.* $n \cdot 3^x + y = 2^z$ *then the element* n *is reachable.*

Proof. If the following formulas are valid in the structure $\mathfrak{M}$
$\{q := 0;$ **while** $q \neq x$ **do** $q := q + 1$ **od**$\}(x = q)$,
$\{q := 0;$ **while** $q \neq y$ **do** $q := q + 1$ **od**$\}(y = q)$,
$\{q := 0;$ **while** $q \neq z$ **do** $q := q + 1$ **od**$\}(z = q)$
and the following equation is valid too $P3(n, x) + y = P2(z)$ then it is easy to verify that the formula $\{t := 0;$ **while** $n \neq t$ **do** $t := t + 1$ **od**$\}(t = n)$ is valid too.

Nr	Reason
1	a1=P2(z) is reachable
2	y+a2 =a1, a2 is reachable and a2=2^z-y
3	a3=P3(1,x) is reachable , a3=3^x
4	$\left\{\begin{array}{l} q := 1; a5 := a3; \\ \textbf{while } a5 \neq a2 \textbf{ do} \\ \quad q := q + 1; \\ \quad a5 := a5 + a3 \\ \textbf{od} \end{array}\right\}(q * a3 = a2)$ hence q=n

■

6.3 An introduction to calculus of programs $\mathcal{AL}$

For the convenience of the reader we cite the axioms and inference rules of calculus of programs i.e. algorithmic logic $\mathcal{AL}$.
Note. *Every axiom of algorihmic logic is a tautology. Every inference rule of $\mathcal{AL}$ is sound.* [MS87]

Axioms

axioms of propositional calculus

Ax_1 $((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \delta) \Rightarrow (\alpha \Rightarrow \delta)))$

Ax_2 $(\alpha \Rightarrow (\alpha \vee \beta))$

Ax_3 $(\beta \Rightarrow (\alpha \vee \beta))$

Ax_4 $((\alpha \Rightarrow \delta) \Rightarrow ((\beta \Rightarrow \delta) \Rightarrow ((\alpha \vee \beta) \Rightarrow \delta)))$

Ax_5 $((\alpha \wedge \beta) \Rightarrow \alpha)$

Ax_6 $((\alpha \wedge \beta) \Rightarrow \beta)$

Ax_7 $((\delta \Rightarrow \alpha) \Rightarrow ((\delta \Rightarrow \beta) \Rightarrow (\delta \Rightarrow (\alpha \wedge \beta))))$

Ax_8 $((\alpha \Rightarrow (\beta \Rightarrow \delta)) \Leftrightarrow ((\alpha \wedge \beta) \Rightarrow \delta))$

Ax_9 $((\alpha \wedge \neg\alpha) \Rightarrow \beta)$

Ax_{10} $((\alpha \Rightarrow (\alpha \wedge \neg\alpha)) \Rightarrow \neg\alpha)$

Ax_{11} $(\alpha \vee \neg\alpha)$

axioms of predicate calculus

Ax_{12} $((\forall x)\alpha(x) \Rightarrow \alpha(x/\tau)))$
where term τ is of the same type as the variable x

Ax_{13} $(\forall x)\alpha(x) \Leftrightarrow \neg(\exists x)\neg\alpha(x)$

axioms of calculus of programs

Ax_{14} $K((\exists x)\alpha(x)) \Leftrightarrow (\exists y)(K\alpha(x/y))$ for $y \notin V(K)$

Ax_{15} $K(\alpha \vee \beta) \Leftrightarrow ((K\alpha) \vee (K\beta))$

Ax_{16} $K(\alpha \wedge \beta) \Leftrightarrow ((K\alpha) \wedge (K\beta))$

Ax_{17} $K(\neg\alpha) \Rightarrow \neg(K\alpha)$

Ax_{18} $((x := \tau)\gamma \Leftrightarrow (\gamma(x/\tau) \wedge (x := \tau)true)) \wedge ((q := \gamma\prime)\gamma \Leftrightarrow \gamma(q/\gamma\prime))$

Ax_{19} $\textbf{begin } K; M \textbf{ end } \alpha \Leftrightarrow K(M\alpha)$

Ax_{20} $\textbf{if } \gamma \textbf{ then } K \textbf{ else } M \textbf{ fi } \alpha \Leftrightarrow ((\neg\gamma \wedge M\alpha) \vee (\gamma \wedge K\alpha))$

Ax_{21} $\textbf{while } \gamma \textbf{ do } K \textbf{ od } \alpha \Leftrightarrow ((\neg\gamma \wedge \alpha) \vee (\gamma \wedge K(\textbf{while } \gamma \textbf{ do } K \textbf{ od}(\neg\gamma \wedge \alpha))))$

Ax_{22} $\bigcap K\alpha \Leftrightarrow (\alpha \wedge (K \bigcap K\alpha))$

Ax_{23} $\bigcup K\alpha \equiv (\alpha \vee (K \bigcup K\alpha))$

Inference rules

propositional calculus

R_1 $\dfrac{\alpha, (\alpha \Rightarrow \beta)}{\beta}$ (also known as modus ponens) *predicate calculus*

$$R_6 \qquad \frac{(\alpha(x) \ \Rightarrow \ \beta)}{((\exists x)\alpha(x) \ \Rightarrow \ \beta)}$$

$$R_7 \qquad \frac{(\beta \ \Rightarrow \ \alpha(x))}{(\beta \Rightarrow (\forall x)\alpha(x))}$$

calculus of programs AL

$$R_2 \qquad \frac{(\alpha \Rightarrow \beta)}{(K\alpha \Rightarrow K\beta)}$$

$$R_3 \qquad \frac{\{s(\mathbf{if}\ \gamma\ \mathbf{then}\ K\ \mathbf{fi})^i(\neg\gamma \wedge \alpha) \Rightarrow \beta\}_{i \in N}}{(s(\mathbf{while}\ \gamma\ \mathbf{do}\ K\ \mathbf{od}\ \alpha) \Rightarrow \beta)}$$

$$R_4 \qquad \frac{\{(K^i\alpha \Rightarrow \beta)\}_{i \in N}}{(\bigcup K\alpha \Rightarrow \beta)}$$

$$R_5 \qquad \frac{\{(\alpha \Rightarrow K^i\beta)\}_{i \in N}}{(\alpha \Rightarrow \bigcap K\beta)}$$

In rules R_6 and R_7, it is assumed that x is a variable which is not free in β, i.e. $x \notin FV(\beta)$. The rules are known as the rule for introducing an existential quantifier into the antecedent of an implication and the rule for introducing a universal quantifier into the successor of an implication. The rules R_4 and R_5 are algorithmic counterparts of rules R_6 and R_7. They are of a different character, however, since their sets of premises are infinite. The rule R_3 for introducing a **while** into the antecedent of an implication of a similar nature. These three rules are called ω-rules. The rule R_1 is known as *modus ponens*, or the *cut*-rule. In all the above schemes of axioms and inference rules, α, β, δ are arbitrary formulas, γ and γ' are arbitrary open formulas, τ is an arbitrary term, s is a finite sequence of assignment instructions, and K and M are arbitrary programs.

Theorem 38 (*theorem on completeness of the calculus $\mathcal{AL}$*) *Let $\mathcal{T} = \langle \mathcal{L}, \mathcal{C}, \mathcal{A}x \rangle$ be a consistent algorithmic theory, let $\alpha \in \mathcal{L}$ be a formula. The following conditions are equivalent*

(i) Formula α is a theorem of the theory T, $\alpha \in \mathcal{C}(\mathcal{A}x)$,

(ii) Formula α is valid in every model of the theory T, $\mathcal{A}x \models \alpha$.

The proof may be found in [MS87].

6.4 An introduction to algorithmic theory of natural numbers $\mathcal{ATN}$

The language of algorithmic theory of natural numbers $\mathcal{ATN}$ is very simple. Its alphabet contains one constant 0 *zero* , one one-argument functor s and predicate $=$ of equality. We shall write $x+1$ instead of $s(x)$. Axioms of $\mathcal{ATN}$ were presented in the book [MS87]

$$A_1)\ \forall x\{q := 0;\ \textbf{while}\ q \neq x\ \textbf{do}\ q := s(q)\ \textbf{od}\}(q = x) \qquad (R)$$

$$A_2)\ \forall x\ s(x) \neq 0 \qquad (N)$$

$$A_3)\ \forall x \forall y\, s(x) = s(y) \implies x = y \qquad (J)$$

We can add another two-argument functor $+$ and its definition

$$A_4)\ \forall x\, \forall y \left\{ \begin{array}{l} q := 0; w := x; \\ \textbf{while}\ q \neq y\ \textbf{do} \\ \quad q := s(q)\ ; w := s(w) \\ textbf od \end{array} \right\} (x + y = w) \quad (D)$$

The termination property of the program in A_4 is a theorem of $\mathcal{ATN}$ theory as well as the formulas $x + 0 = x$ and $x + s(y) = s(x + y)$.

*A sample (8 – 12) of **Theorems** of $\mathcal{ATN}$*

$$\mathcal{ATN} \vdash \exists_x\, \alpha(x) \Leftrightarrow \{x := 0\} \bigcup \{x := x + 1\}\alpha(x) \qquad (8)$$

$$\mathcal{ATN} \vdash \forall_x\, \alpha(x) \Leftrightarrow \{x := 0\} \bigcap \{x := x + 1\}\alpha(x) \qquad (9)$$

Law of Archimedes

$$\mathcal{ATN} \vdash 0 < x < y \implies \{a := x;\ \textbf{while}\ a < y\ \textbf{do}\ a := a + x\ \textbf{od}\}(a \geq y) \tag{10}$$

Scheme of induction

$$\mathcal{ATN} \vdash \Big(\alpha(x/0) \wedge \forall_x \big(\alpha(x) \Rightarrow \alpha(x/s(x))\big)\Big) \implies \forall_x \alpha(x) \tag{11}$$

Correctness of Euclid's algorithm

$$\mathcal{ATN} \vdash \begin{pmatrix} n_0 > 0 \wedge \\ m_0 > 0 \end{pmatrix} \implies \left\{ \begin{array}{l} n := n_0;\ m := m_0; \\ \textbf{while}\, n \neq m\, \textbf{do} \\ \quad \textbf{if}\, n > m\ \textbf{then}\ \ n := n \dot{-} m \\ \qquad \textbf{else}\ \ m := m \dot{-} n \\ \quad \textbf{fi} \\ \textbf{od} \end{array} \right\} (n = gcd(n_0, m_0)) \tag{12}$$

The theory $\mathcal{ATN}$ enjoys an important property of categoricity.

Theorem 39 (*meta*-theorem on categoricity of $\mathcal{ATN}$) *Every model $\mathfrak{A}$ of the algorithmic theory of natural numbers is isomorphic to the structure $\mathfrak{N}$, c.f. subsection 6.1.*

6.5 Proof of lemma 4

Let P and P' be two programs. Let α be any formula. The semantic property ***programs P and P' are equivalent with respect to the postcondition*** α is expressed by the formula of the form $(\{P\}\,\alpha \Leftrightarrow \{P'\}\,\alpha)$. We shall use the following tautology of calculus of programs $\mathcal{AL}$.

$$\vdash \left(\overbrace{\left\{ \begin{array}{l} \textbf{while}\ \gamma\ \textbf{do} \\ \quad \textbf{if}\ \delta\ \textbf{then}\ K\ \textbf{else}\ M\ \textbf{fi} \\ \textbf{od}; \end{array} \right\}}^{P:} \alpha \Leftrightarrow \overbrace{\left\{ \begin{array}{l} \textbf{while}\ \gamma\ \textbf{do} \\ \quad \textbf{while}\ \gamma \wedge \delta\ \textbf{do}\ K\ \textbf{od}; \\ \quad \textbf{while}\ \gamma \wedge \neg\delta\ \textbf{do}\ M\ \textbf{od} \\ \textbf{od} \end{array} \right\}}^{P':} \alpha \right) \tag{13}$$

We apply the axioms Ax20 and Ax21

$$\vdash \left(\left\{ \begin{array}{l} \textbf{while}\ \gamma\ \textbf{do} \\ \quad \textbf{if}\ \delta\ \textbf{then}\ K\ \textbf{else}\ M\ \textbf{fi} \\ \textbf{od}; \end{array} \right\} \alpha \Leftrightarrow \left\{ \begin{array}{l} \textbf{if}\ \gamma\ \textbf{then} \\ \quad \textbf{while}\ \gamma \wedge \delta\ \textbf{do}\ K\ \textbf{od}; \\ \quad \textbf{while}\ \gamma \wedge \neg\delta\ \textbf{do}\ M\ \textbf{od}\ ; \\ \quad \textbf{while}\ \gamma\ \textbf{do} \\ \qquad \textbf{while}\ \gamma \wedge \delta\ \textbf{do}\ K\ \textbf{od}; \\ \qquad \textbf{while}\ \gamma \wedge \neg\delta\ \textbf{do}\ M\ \textbf{od} \\ \quad\ \ \textbf{od} \\ \textbf{fi} \end{array} \right\} \alpha \right) \tag{14}$$

We can omit the instruction **if** (why?) . We swap internal instructions **while** inside the instruction **while**.

$$\vdash \left(\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{if } \delta \textbf{ then } K \textbf{ else } M \textbf{ fi} \\ \textbf{od}; \end{array} \right\} \alpha \Leftrightarrow \left\{ \begin{array}{l} \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od } ; \\ \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od} \\ \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od}; \\ \quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od} \\ \textbf{od} \end{array} \right\} \alpha \right) \tag{15}$$

We can safely skip the second instruction while.

$$\vdash \left(\left\{ \begin{array}{l} \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{if } \delta \textbf{ then } K \textbf{ else } M \textbf{ fi} \\ \textbf{od}; \end{array} \right\} \alpha \Leftrightarrow \left\{ \begin{array}{l} \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od } ; \\ \textbf{while } \gamma \textbf{ do} \\ \quad \textbf{while } \gamma \wedge \neg\delta \textbf{ do } M \textbf{ od}; \\ \quad \textbf{while } \gamma \wedge \delta \textbf{ do } K \textbf{ od} \\ \textbf{od} \end{array} \right\} \alpha \right) \tag{16}$$

Now, we put $(\gamma \stackrel{df}{=} n \neq 1)$, $(\delta \stackrel{df}{=} even(n))$, $(K \stackrel{df}{=} \{n := n \div 2\})$ and $(M \stackrel{df}{=} \{n := 3n+1\})$. We make use of two theorems $(even(n) \implies n \neq 1)$ and $(M\delta)$ of theory $\mathcal{AP}$ c.f. subsection 6.2.

$$\mathcal{T} \vdash \left(\left\{ \begin{array}{l} \textbf{while } n \neq 1 \textbf{ do} \\ \quad \boxed{\begin{array}{l} \textbf{if } even(n) \\ \textbf{then } n := n \div 2 \\ \textbf{else } n := 3n+1 \\ \textbf{fi} \end{array}} \\ \textbf{od}; \end{array} \right\} (n=1) \Leftrightarrow \left\{ \begin{array}{l} \textbf{while } even(n) \textbf{ do } n := n \div 2 \textbf{ od}; \\ \textbf{while } n \neq 1 \textbf{ do} \\ \quad \boxed{\begin{array}{l} n := 3n+1; \\ \textbf{while } \ even(n) \textbf{ do } n := n \div 2 \textbf{ od} \end{array}} \\ \textbf{od} \end{array} \right\} (n=1) \right). \tag{17}$$

6.6 *Proof of invariant of algorithm Gr3*

We are going to prove the following implication (19) is a theorem of algorithmic theory of natural numbers $\mathcal{ATN}$.

$$\mathcal{ATN} \vdash (\varphi \implies \{\Delta_3\}\varphi) \tag{18}$$

$$\overbrace{\left(\begin{array}{l} n \cdot 3^i + Y_i = m_i \cdot 2^{Z_i} \wedge \\ Z_i = \sum_{j=0}^{i} k_j \wedge X_i = i \wedge \\ Y_i = \sum_{j=0}^{i-1} \left(3^{i-1-j} \ . \ 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq i} \left(m_l = m_{l-1}/2^{k_l} \wedge \right. \\ \left. k_l = exp(m_l, 2) \right) \end{array} \right)}^{\varphi} \implies \overbrace{\left\{ \begin{array}{l} aux := 3 * m_i + 1; \\ k_{i+1} := exp(aux, 2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ Y_{i+1} := 3Y_i + 2^{Z_i}; \\ Z_{i+1} := Z_i + k_{i+1}; \\ X_i := i; ; \\ \hline i := i+1; \end{array} \right\}}^{\Delta_3} \overbrace{\left(\begin{array}{l} n \cdot 3^i + Y_i = m_i \cdot 2^{Z_i} \wedge \\ Z_i = \sum_{j=0}^{i} k_j \wedge \wedge X_i = i \\ Y_i = \sum_{j=0}^{i-1} \left(3^{i-1-j} \ . \ 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq i} \left(m_l = m_{l-1}/2^{k_l} \wedge \right. \\ \left. k_l = exp(m_l, 2) \right) \end{array} \right)}^{\varphi} \tag{19}$$

We apply the axiom of assignment instruction Ax_{18}. Note, we applied also axiom Ax_{19} of composed instruction. Namely, in the implication (19) we replace the its successor $\{\Delta_3\}\varphi$ by the formula $\{\Delta_3^{(1)}\}\varphi^{(1)}$.

$$\varphi \implies \overbrace{\left\{\begin{array}{c} aux := 3 * m_i + 1; \\ k_{i+1} := exp(aux, 2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ Y_{i+1} := 3Y_i + 2^{Z_i}; \\ Z_{i+1} := Z_i + k_{i+1}; \\ \hline X_i := i;; \end{array}\right\}}^{\Delta_3^{(1)}} \overbrace{\left(\begin{array}{l} n \cdot 3^{(i+1)} + Y_{(i+1)} = m_{(i+1)} \cdot 2^{Z_{(i+1)}} \wedge \\ Z_{(i+1)} = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ Y_{(i+1)} = \sum_{j=0}^{i} \Big(3^{(i+1)-1-j} \ . \ 2^{Z_j} \Big) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \big) \end{array}\right)}^{\varphi^{(1)}} \tag{20}$$

$$\varphi \implies \left\{\begin{array}{c} aux := 3 * m_i + 1; \\ k_{i+1} := exp(aux, 2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ Y_{i+1} := 3Y_i + 2^{Z_i}; \\ \hline Z_{i+1} := Z_i + k_{i+1}; \end{array}\right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + Y_{(i+1)} = m_{(i+1)} \cdot 2^{Z_{(i+1)}} \wedge \\ Z_{(i+1)} = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ Y_{(i+1)} = \sum_{j=0}^{i} \Big(3^{(i+1)-1-j} \ . \ 2^{Z_j} \Big) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \big) \wedge \end{array}\right) \tag{21}$$

$$\varphi \implies \left\{\begin{array}{c} aux := 3 * m_i + 1; \\ k_{i+1} := exp(aux, 2); \\ m_{i+1} := aux/2^{k_{i+1}}; \\ \hline Y_{i+1} := 3Y_i + 2^{Z_i}; \end{array}\right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + Y_{(i+1)} = m_{(i+1)} \cdot 2^{(Z_i + k_{i+1})} \wedge \\ (Z_i + k_{i+1}) = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ Y_{(i+1)} = \sum_{j=0}^{i} \Big(3^{(i+1)-1-j} \ . \ 2^{Z_j} \Big) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \big) \end{array}\right) \tag{22}$$

$$\varphi \implies \left\{\begin{array}{c} aux := 3 * m_i + 1; \\ k_{i+1} := exp(aux, 2); \\ \hline m_{i+1} := aux/2^{k_{i+1}}; \end{array}\right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = m_{(i+1)} \cdot 2^{(Z_i + k_{i+1})} \wedge \\ (Z_i + k_{i+1}) = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum_{j=0}^{i} \Big(3^{(i+1)-1-j} \ . \ 2^{Z_j} \Big) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \big) \end{array}\right) \tag{23}$$

$$\varphi \implies \left\{\begin{array}{c} aux := 3 * m_i + 1; \\ \hline k_{i+1} := exp(aux, 2); \end{array}\right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = (aux/2^{k_{i+1}}) \cdot 2^{(Z_i + k_{i+1})} \wedge \\ (Z_i + k_{i+1}) = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum_{j=0}^{i} \Big(3^{(i+1)-1-j} \ . \ 2^{Z_j} \Big) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \big(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \big) \end{array}\right) \tag{24}$$

$$\varphi \implies \left\{ \ aux := 3 * m_i + 1; \right\} \left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = (aux/2^{(exp(aux,2)))} \cdot 2^{(Z_i+(exp(aux,2)))} \wedge \\ (Z_i + (exp(aux,2))) = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum_{j=0}^{i} \left(3^{(i+1)-1-j} \cdot 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \left(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \right) \end{array} \right) \tag{25}$$

$$\varphi \implies \{\} \overbrace{\left(\begin{array}{l} n \cdot 3^{(i+1)} + (3Y_i + 2^{Z_i}) = ((3 * m_i + 1)/2^{(exp((3*m_i+1),2)))} \cdot 2^{(Z_i+(exp((3*m_i+1),2)))} \wedge \\ (Z_i + (exp((3 * m_i + 1), 2))) = \sum_{j=0}^{(i+1)} k_j \wedge X_{i+1} = (i+1) \wedge \\ (3Y_i + 2^{Z_i}) = \sum_{j=0}^{i} \left(3^{(i+1)-1-j} \cdot 2^{Z_j} \right) \wedge \\ m_0 = n \wedge \ k_0 = exp(m_0, 2) \wedge \\ \forall_{0<l\leq(i+1)} \left(m_l = m_{l-1}/2^{k_l} \wedge k_l = exp(m_l, 2) \right) \end{array} \right)}^{\psi} \tag{26}$$

The implications (19) –(26) are mutually equivalent.
One can easily verify that the last implication ($\varphi \implies \psi$) (26) is a theorem of Presburger arithmetic $\mathcal{T}$ and hence it is a theorem of $\mathcal{ATN}$ theory.
Therefore the first implication ($\varphi \implies \{\Delta_3\}\varphi$) (19) is a theorem of algorithmic theory of natural numbers $\mathcal{ATN}$.

q.e.d.

Remark 40 *Note, the process of creation a proof like this can be automatized. The verification of the above proof can be automatized too.*

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