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ON CERTAIN PROPERTY NOT EXPRESSIBLE IN PAL

Grażyna M i r k o w s k a
University of Warsaw

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A b s t r a c t. The property of finite degree of nondeterminism is not expressible by a formula and by any set of formulas in propositional algorithmic logic PAL.

K e y w o r d s : logic of programs, expressiveness, degree of nondeterminism.

INTRODUCTION

We shall consider the propositional algorithmic logic PAL. The important and difficult problem of complete axiomatization is as follows: find a set of axioms and rules of inference such that the notions of syntactical consequence operation and of semantical consequence operation coincide. The problem has been solved in [5] but the logical system presented there has two disadvantages. Firstly, the set of inference rules contains ω-rule and this cannot be improved since of noncompactness. Secondly, we ought to restrict the semantics to the class of structures with so called finite degree of nondeterminism property. Since we had not any syntactical characterization of this notion then to prove completeness we had to replace it by a stronger assumption of bounded degree of nondeterminism which is axiomatizable.

The present paper shows that we can not improve the situation, since the finite degree of nondeterminism property is not expressible in propositional algorithmic logic PAL.

The paper results from a discussion with D.Kozen. I wish to thank him for collaboration.

1. PRELIMINARY NOTIONS

Let r be a binary relation in a nonempty set X. We shall say that r has a finite degree of nondeterminism property, for short, fdn property iff for every element $x \in X$ the set $\{x:xrx\}$ is finite. Can we express this property by a formula or by a set of formulas of propositional algorithmic logic PAL ? We shall start with a short explanation of syntax and semantics of the considered system (see [4], [5], [6] for more details).

The set of formulas F contains all formulas of classical propositional calculus ,build from the set of propositional variables V_o by means of logical functors $\vee$, $\wedge$, $\Rightarrow$, $\sim$ and of all algorithmic formulas of the form $\Box M\beta$ and $\Diamond M\beta$, where β is an arbitrary formula and M is an arbitrary program in this language.

The set $\mathcal{T}$ of programs is constructed from the set V_p of atoms (or program variables) by means of the following rule : if β is a classical formula and M ,M′ are programs then the expressions

begin M; M′ end,
if β then M else M′ fi,
while β do M od,
either M or M′ ro

are programs.

The semantics of the language is described by means of Kripke-like structure

$$\mathfrak{M} = \langle S, I, w \rangle,$$

where S is a set of states, I is an interpretation of the program variables which to every program variable K assigns a binary relation $K_{\mathfrak{M}}$ in S ,and w is a mapping which to every propositional variable assigns a subset of S.

For a given structure $\mathfrak{M}$ and a given state s we define the satisfiability relation as usual for classical formulas and for algorithmic formulas as follows.

$\mathfrak{M}, s \models \Diamond M\beta$ iff there exists a successful computation of the program M in the structure $\mathfrak{M}$ starting from the state s such that its result satisfies the formula β .

$\mathfrak{M}, s \models \Box M\beta$ iff all computations of the program M starting from the state s in the structure $\mathfrak{M}$ are successful and all its results satisfy the formula β.

We shall say that the structure $\mathfrak{M}$ has fdn property iff for every $K \in V_p$ the binary relation $K_{\mathfrak{M}}$ has finite degree of nondeterminism property, i.e. $\mathrm{card}\{s' : (s,s') \in K_{\mathfrak{M}}\} < \aleph_0$ for all $s \in S$.

2. THE MAIN RESULT

We shall formulate the problem of the paper as follows: is there a set Z of formulas of PAL such that for every structure $\mathfrak{M}$, $\mathfrak{M} \models Z$ iff $\mathfrak{M}$ has fdn property ?

Lemma

Does not exists a formula β such that for every structure $\mathfrak{M}$, $\mathfrak{M} \models \beta$ iff $\mathfrak{M}$ has fdn property.

Proof.

Let us suppose, on the contrary, that for some formula β,

(1) $\mathfrak{M} \models \beta$ iff $\mathfrak{M}$ has fdn property,

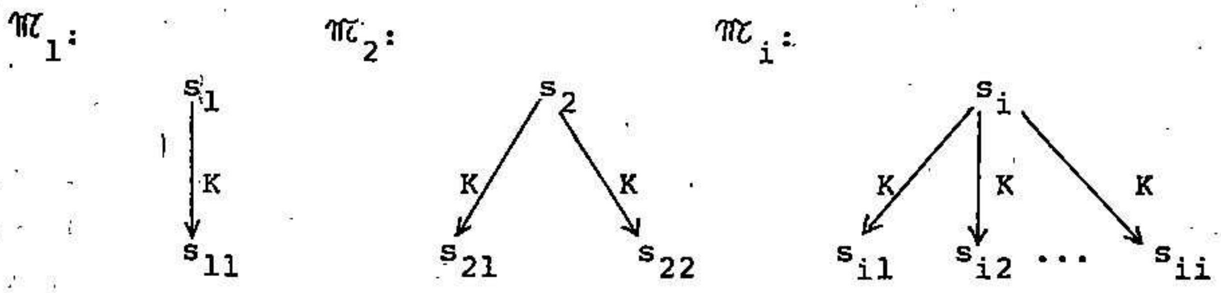
for every structure $\mathfrak{M}$.

Let us consider the family of structures $\{\mathfrak{M}_i\}_{i \in N}$ such that $\mathfrak{M}_i = \langle S_i, I_i, w_i \rangle$, where

$S_i = \{s_i, s_{i1}, \ldots, s_{ii}\}$,

$I_i(K) = \{(s_i, s_{ij}) : j \leq i\}$ and $I_i(K') = \emptyset$ for every $K' \neq K$, $K' \in V_p$,

$w_i(q) = S_i$ for all propositional variables $q \in V_o$.



FINAL REMARKS

1. It is well known that in general the ultraproduct theorem does not hold in PAL due to the noncompactness (cf. [5]).

2. For every natural number m, the property of the structure $\mathfrak{M}$ called mth degree of nondeterminism property (i.e.the property $(\forall s\in\mathfrak{M})(\forall K\in V_p)$ card$\{s' : s\ K_{\mathfrak{M}}\ s'\}\leq m$)
is expressible by a single scheme of a formula in PAl in contrast to fdn property (cf. [5]).

3. Similar results can be formulated and proved for propositional dynamic logic PDL [1] . The fdn property is not expressible in PDL .

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