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Dombrova Research

Main points of the proof

a guide on the longer paper "On Collatz theorem II"

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Summary

We got some remarks: "this paper is too long". *Vox populi vox dei.* We present this short, informal text hoping that it will serve you as a guide . Note, all references to a page are to the longer text.

Four main points of our argumentation are:

T0. The nature of Collatz problem is algorithmic, it concerns the halting property of Collatz algorithm Cl, see p. 2. I.e. the Collatz conjecture reduces to proving the halting property of algorithm Cl.

T1. FACT. There exist infinite Collatz computations.
Namely, the elementary theory of addition of natural numbers $\mathcal{AP}$, p. 24, also known as Presburger arithmetic has a non-standard model $\mathfrak{M}$, p.20-23. The universe of the structure $\mathfrak{M}$ contains the standard natural numbers and other non-standard, unreachable elements.
THM. 1. For every unreachable element x its Collatz computation is infinite.

T2. FACT. Every element n determines an infinite sequence $\{k_l\}$ of elements, see recurrence (rec1) page.2,

THM. 2. For every element n, the following conditions are equivalent

(*i*) Collatz computation for n is finite,

(*ii*) there exists the least triple $\langle x, y, z\rangle$ such that the condition ($\star$) holds

$$x = (\mu i)\left((n \cdot 3^i + y = 2^z) \wedge \left(y = \sum_{j=0}^{i-1} \left(3^{i-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)\right) \wedge \left(z = \sum_{l=0}^{i} k_l\right)\right). \qquad (\star)$$

T3. THM. 3. If the Collatz computation of an element n is infinite, then the element is unreachable.

1.1. One can limit her/his considerations on Collatz conjecture to a structure of the following *signature* $\langle U; +, 0, 1, =\rangle$. For the operation of multiplication is **not needed**.
(Note, programmers may prefer the word *interface* instead of signature.)

1.2. We shall consider models of Presburger arithmetic of addition of natural numbers. There are two computable models of Presburger arithmetic: one is standard set of natural numbers $\mathfrak{N} = \langle N;\, 0, 1, +, =\rangle$, the second model is the subset$M$ of the set of complex numbers that consists of all numbers $\boxed{a + b\,\imath}$ such that $a \in Z$ (a is integer) and $b \in Q^+$ (b is a non-negative rational number). Hence $\mathfrak{M} = \langle M;\, 0, 1, +, =\rangle$, c.f. subsection 7.1 pages 20-23 . Note, $N \subsetneq M$.

1.4. The section 2, page 4, exhibits an *infinite* computation of Collatz algorithm. Therefore, the Collatz conjecture is not a theorem of Presburger arithmetic (nor Peano arithmetic).

1.5. Moreover, the *Collatz conjecture can not be expressed by any first-order formula*. For the Presburger arithmetic is a complete and decidable theory.

1.6. Every element n designates its unique Collatz computation.
The whole computation consists of either standard, reachable numbers or it contains non-standard, unreachable elements only. This is easily seen from the table 1 on page 21.

2.1. The following recurrence

$$\left.\begin{array}{ll} k_0 = exp(n,2) \quad \wedge \quad m_0 = \frac{n}{2^{k_0}} & \\ k_{i+1} = exp(3m_i+1, 2) \quad \wedge \quad m_{i+1} = \frac{3m_i+1}{2^{k_{i+1}}} & \text{for } i \geq 0 \end{array}\right\} \qquad \text{(rec2)}$$

replaces original Collatz recurrence (rec1) , p. 2. Both recurrences designate infinite sequences.The second sequence $\{m_i\}$ consists of all odd numbers that appear in the Collatz computation for n. It is the sequence $\{k_i\}$ that plays an important role in further considerations.

2.2. Programs Cl and Gr are equivalent, see Lemma 4.1, page 8,

2.3. Program $Gr1$ is an extension of program Gr and one can check, that it calculates the sequence described in point 2.1.

2.4. Programs $Gr2$ and $Gr3$ show more properties of the Collatz computations.
The formula $n \cdot 3^x + y = m_i \cdot 2^z$ is an invariant of the program $Gr2$.
Additionally, program $Gr3$ shows an increasing sequence of triples $\langle i, Y_i, Z_i\rangle$ where $Y_0 = 0$ and $Y_{i+1} = 3Y_i + 2^{Z_i}$ and $Z_{i+1} = Z_i + k_{i+1}$ are consecutive values of variables x, y, z used in program $Gr2$.
Any of four programs $Gr, Gr1, Gr2, Gr3$ halts, if and only if, the Collatz algorithm Cl halts.
I.e. the computation of Collatz algorithm for a given element n is finite, if and only if, there exists a standard number i, such that $m_i = 1$.
This in turn, holds iff i is the least triple $\langle i, y, z\rangle$ such that $n \cdot 3^i + y = 2^z$. One can verify that it happens iff

$$n \cdot 3^i + \left(\sum_{j=0}^{i-1} \left(3^{i-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right) \right) = 2^{\sum_{l=0}^{i} k_l}$$

2.5. In this way we prove the point 2.

2.6. The Figure 1 , states the same as point 2, see Figure 5 p. 17.

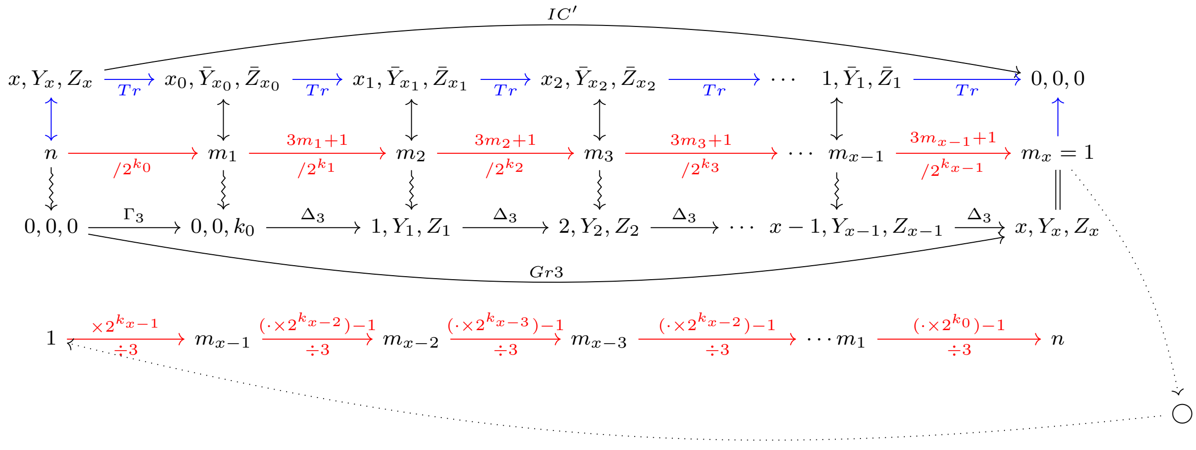


Figure 1. CASE OF FINITE COMPUTATION ILLUSTRATED
Middle row, (with red arrows) represents computation of $Gr1$, elements k_i and m_i are given by the recurrence (rec2)
third row shows computation of $Gr3$, the subsequent triples are $X_{i+1} = i+1, Y_{i+1} = 3Y_i + 2^Z_i, Z_{i+1} = Z_i + k_i$
first row (blue arrows) shows computation of algorithm IC on triples, $\bar{Y}_x = Y_x$ and $\bar{Z}_x = Z_x$ and for $i = x, \dots, 1$ we have $\bar{Z}_{i-1} = \bar{Z}_i - k_i$ and $\bar{Y}_{i-1} = (\bar{Y}_i / 2^{k_i}) - 3^{i-1}$

3.1. To complete the proof, one needs to show, that if for a certain element n_0 its Collatz computation is infinite and if $n_0 \cdot 3^x + y = 2^z$, then n_0 is a non-standard element.
Noe, that the backward computation that begins with triple $\langle x, y, z \rangle$ is *Error*-free.

3.2. A computation of the Collatz algorithm Cl is infinite iff the computation of algorithm Gr is infinite. (For the programs Cl and Gr are equivalent.)

$$m_0 \xrightarrow[/2^{k_0}]{} m_1 \xrightarrow[/2^{k_1}]{3m_1+1} m_2 \xrightarrow[/2^{k_2}]{3m_2+1} m_3 \xrightarrow[/2^{k_3}]{3m_3+1} \cdots \; m_{x-1} \xrightarrow[/2^{k_{x-1}}]{3m_{x-1}+1} m_x \neq 1 \qquad \cdots \; \perp$$

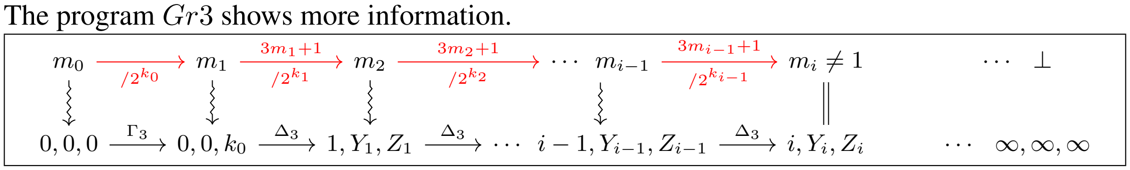
For every natural number $i \in N$ the value of m_i differs from 1, $\forall_{i \in N}\, m_i \neq 1$.

3.3. We recall that the formula

$$\forall_{i \in N}\, n \cdot 3^x + y = m_i \cdot 2^z$$

is invariant of the program $Gr2$.

3.4. The program $Gr3$ shows more information.



We remark that the values of X_i, Y_i, Z_i do form an infinite, increasing sequence of triples

$$\langle X_{i-1}, Y_{i-1}, Z_{i-1}\rangle \prec \langle X_i, Y_i, Z_i\rangle.$$

This remark is justified by following recurrences

$$X_i = i \quad \& \quad Z_i = Z_{i-1} + k_i \quad \& \quad Y_i = 3Y_{i-1} + 2^{Z_{i-1}}.$$

3.5. Making use of the above equations and of initial values $X_0 = Y_0 = Z_0 = 0$ we can eplace the program $Gr3$ by the following program $Gr4$

$$Gr4: \boxed{\begin{array}{l} \textbf{var}\ n, x, y, z : integer; \\ \quad \Gamma_4 : \boxed{\text{READ}(n);\ x, y, z := 0;} \\ \textbf{while}\ n \cdot 3^x + y \neq 2^z\ \textbf{do} \\ \quad \Delta_4 : \boxed{\begin{array}{l} x := x + 1; \\ y := 3y + 2^z; \\ z := z + k_x; \end{array}} \\ \textbf{od} \end{array}}$$

3.6. Program $Gr4$ terminates iff there exists the number x defined as the least number i such that

$$x \stackrel{df}{=} (\mu i)\Big(n \cdot 3^i + \sum_{j=0}^{i-1} 3^{i-1-j} \cdot 2^{\sum_{l=0}^{j} k_l} = 2^{\sum_{j=0}^{i} k_j}\Big)$$

3.7. On the other hand, it is known, that for every element n, there exist elements $\bar{x}, \bar{y}, \bar{z}$ such, that $n \cdot 3^{\bar{x}} + \bar{y} = 2^{\bar{z}}$.
From the assumption on non-termination, we deduce that element $\bar{x}$ is not a reachable number, unreachable are also elements $\bar{y} = \sum_{j=0}^{\bar{x}-1} 3^{\bar{x}-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}$ and $\bar{z} = 2^{\sum_{j=0}^{\bar{x}} k_j}$.

3.8. Now, we apply the lemma 7.6 and obtain that the number n is also unreachable.

3.9. Let the Collatz computation for an element n be infinite, and a triple $\langle x', y', z'\rangle$ represents the element n. There exists another triple $\langle x", y", z"\rangle$ that represents the same element n and is lesser $\langle x", y", z"\rangle \prec \langle x', y', z'\rangle$.
In order to verify this, it suffices to put $x" = x' - 1$ & $y" = \frac{1}{3} \cdot (y' + 2^{z'+1})$ & $z" = z'$.

3.10. We illustrate our consideration by the following Figure 2.

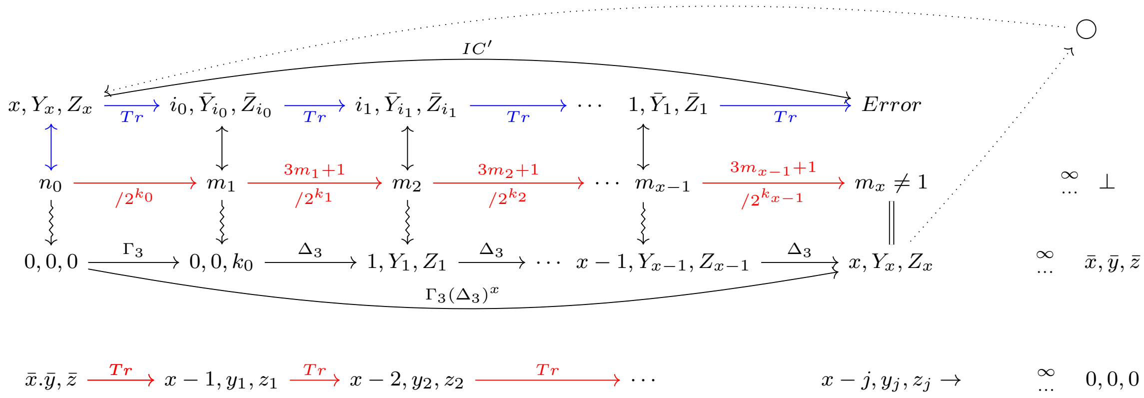


Figure 2. CASE OF INFINITE COMPUTATION
Middle row, (with red arrows) represents computation of $Gr1$, elements k_i and m_i are given by the recurrence (rec2)
third row shows computation of $Gr3$, the subsequent triples are $X_{i+1} = i+1, Y_{i+1} = 3Y_i + 2^Z_i, Z_{i+1} = Z_i + k_i$
first row (blue arrows) shows computation of algorithm IC on triples, $\bar{Y}_x = Y_x$ and $\bar{Z}_x = Z_x$ and for $i = x, \ldots, 1$ we have $\bar{Z}_{i-1} = \bar{Z}_i - k_i$ and $\bar{Y}_{i-1} = (\bar{Y}_i/2^{k_i}) - 3^{i-1}$

Historical remarks

Pàl Erdős said on Collatz conjecture: *"Mathematics may not be ready for such problems."* Let's see.

- Mojżesz Presburger has proved the completeness and decidability of arithmetic of addition of natural nubers in 1929.
- In the same year Stanisław Jaśkowski found a non-standard model of Presburger theory (see a note of A. Tarski of 1934) .
- Kurt Gödel (1931) published his theorem on incompleteness of Peano arithmetic.
- Thoralf Skolem (in 1934) wrote a paper on the non-characterization of the series of numbers by means of a finite or countably infinite number of statements with exclusively individual variables *Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschließlich Zahlenvariablen* , Fundamenta Mathematicae, ,**23**,1, 150–161, `http://matwbn.icm.edu.pl/ksiazki/fm/fm23/fm23115.pdf`
- Stephen Kleene shown (in 1936) that any recurrence defining a computable function can be replaced by the operation of effective minimum (nowadays one can say every recursive function in the integers, is programmable by means of **while** instruction).
- It seems that P. Erdős was wrong, his colleagues - professors Rozsa Peter and Laszlo Kalmar (specialists in the theory of recursive functions) were able to point it out to him.