Dombrova Research

Main points of the proof

a guide on the longer paper "On Collatz theorem II"

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Summary

We got some remarks: "this paper is (too) long". *Vox populi vox dei.* We present this short, informal text hoping that it will serve you as a guide . Note, all references to a page are to the longer text.
Main points of our argumentation are:

0. The nature of Collatz problem is algorithmic, it concerns the halting property of Collatz algorithm Cl, see p. 2.
   In the previous paper [ ] we have shown that for any given number $n \neq 0$ the Collatz algorithm halts , if and only if, there exists a triple of numbers $\langle x, y, z\rangle$ such that the equality $n \cdot 3^x + y = 2^z$ holds and an algorithm IC starting from the triple halts with $x + y + z = 0$.

1. There exist infinite Collatz computations.
   Namely, the elementary theory of addition of natural numbers $\mathcal{AP}$, p. 24, also known as Presburger arithmetic has a non-standard model $\mathfrak{M}$, p.20-23. The universe of the structure $\mathfrak{M}$ contains the standard natural numbers and other non-standard, unreachable elements.
   For every unreachable element x its Collatz computation is infinite.

2. For every element n, its Collatz computation is finite, if and only if, there exists the least element x such that the following equality holds

$$n \cdot 3^x + \left(\sum_{j=0}^{x-1}\left(3^{x-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)\right) = 2^{\sum_{l=0}^{x} k_l}. \qquad (\mu\ x)$$

Every element n determines an infinite sequence $\{k_l\}$ of elements, see recurrence (rec1), below.

3. We show that, if a Collatz computation of an element n is infinite, then the element is unreachable.

1.1. One can limit her/his considerations on Collatz conjecture to a structure $\mathfrak{M} = \langle M; +, 0, 1, =\rangle$. For the operation of multiplication is **not needed**.

1.2. We shall consider models of Presburger arithmetic of addition of natural numbers. There are two computable (i.e. recursive) models of Presburger arithmetic: one is standard set of natural numbers $\mathfrak{N} = \langle N;\, 0, 1, +, =\rangle$, the second model is the subset$M$ of the set of complex numbers that consists of all numbers $a + \imath b$ such that $a \in Z$ (a is integer) and $b \in Q^+$ (b is a non-negative rational number). Hence $\mathfrak{M} = \langle M;\, 0, 1, +, =\rangle$. subsection 7.1 on page 20 . Note, $N \subsetneq M$.

1.4. The section 2, page 4, exhibits an *infinite* computation of Collatz algorithm. Therefore, the Collatz conjecture is not a theorem of Presburger arithmetic (nor Peano arithmetic). Moreover, the Collatz conjecture can not be expressed by any first-order formula. For the Presburger arithmetic is a complete theory.

1.6. Every element n designates its unique Collatz computation..
The whole computation consists of either standard, reachable numbers or it contains non-standard, unreachable elements only. This is easily seen from the table 1 on page 21.

2.1. The following recurrence

$$\left.\begin{array}{rcl} k_0 = exp(n, 2) & \wedge & m_0 = \frac{n}{2^{k_0}} \\ k_{i+1} = exp(3m_i + 1, 2) & \wedge & m_{i+1} = \frac{3m_i+1}{2^{k_{i+1}}} \qquad \text{for } i \geq 0 \end{array}\right\} \qquad \text{(rec2)}$$

replaces original Collatz recurrence (rec1) , p. 2. Both recurrences designate infinite sequences.The second sequence is a compressed version of the first one.

2.2. Programs Cl and Gr are equivalent.

2.3. Program $Gr1$ is an extension of program Gr and see that it calculates the sequence described in point 2.1.

2.4. Programs $Gr2$ and $Gr3$ show more properties of the Collatz computations.
The formula $n \cdot 3^x + y = m_i \cdot 2^z$ is an invariant of the program $Gr2$.
Namely, program $Gr3$ shows an increasing sequence of triples $\langle i, Y_i, Z_i\rangle$ where $Y_0 = 0$ and $Y_{i+1} = 3Y_i + 2^{Z_i}$ and $Z_{i+1} = Z_i + k_{i+1}$ are consecutive values of variables x, y, z used in program $Gr2$.
All these programs halt if and only if, the Collatz algorithm halts.
I.e. the computation of Collatz algorithm for a given element n is finite, if and only if, there exists a standard number i, such that $m_i = 1$.
This in turn, holds iff i is the least standard number such that

$$n \cdot 3^i + \left(\sum_{j=0}^{i-1}\left(3^{i-1-j} \cdot 2^{\sum_{l=0}^{j} k_l}\right)\right) = 2^{\sum_{l=0}^{i} k_l}$$

2.5. In this way we prove the point 2.

$$
\begin{tikzcd}
x,Y_x,Z_x \arrow[r, "Tr"'] \arrow[d, leftrightarrow] \arrow[rrrrrr, bend left, "IC'"] & x_0,\bar{Y}_{x_0},\bar{Z}_{x_0} \arrow[r, "Tr"'] \arrow[d, leftrightarrow] & x_1,\bar{Y}_{x_1},\bar{Z}_{x_1} \arrow[r, "Tr"'] \arrow[d, leftrightarrow] & x_2,\bar{Y}_{x_2},\bar{Z}_{x_2} \arrow[r, "Tr"'] \arrow[d, leftrightarrow] & \cdots & 1,\bar{Y}_1,\bar{Z}_1 \arrow[r, "Tr"'] \arrow[d, leftrightarrow] & 0,0,0 & \\
n \arrow[r, "/2^{k_0}"'] \arrow[d, squiggly] & m_1 \arrow[r, "3m_1+1", "/2^{k_1}"'] \arrow[d, squiggly] & m_2 \arrow[r, "3m_2+1", "/2^{k_2}"'] \arrow[d, squiggly] & m_3 \arrow[r, "3m_3+1", "/2^{k_3}"'] \arrow[d, squiggly] & \cdots & m_{x-1} \arrow[r, "3m_{x-1}+1", "/2^{k_{x-1}}"'] \arrow[d, squiggly] & m_x = 1 \arrow[u] \arrow[d, equal] \arrow[ddr, bend left] & \\
0,0,0 \arrow[r, "\Gamma_3"] \arrow[rrrrrr, bend right, "Gr3"'] & 0,0,k_0 \arrow[r, "\Delta_3"] & 1,Y_1,Z_1 \arrow[r, "\Delta_3"] & 2,Y_2,Z_2 \arrow[r, "\Delta_3"] & \cdots & x-1,Y_{x-1},Z_{x-1} \arrow[r, "\Delta_3"] & x,Y_x,Z_x & \\
1 \arrow[r, "\times 2^{k_{x-1}}", "\div 3"'] & m_{x-1} \arrow[r, "(\cdot\times 2^{k_{x-2}})-1", "\div 3"'] & m_{x-2} \arrow[r, "(\cdot\times 2^{k_{x-3}})-1", "\div 3"'] & m_{x-3} \arrow[r, "(\cdot\times 2^{k_{x-2}})-1", "\div 3"'] & \cdots m_1 \arrow[rr, "(\cdot\times 2^{k_0})-1", "\div 3"'] & & n & \\
& & & & & & & \bigcirc \arrow[lllllllu, bend left]
\end{tikzcd}
$$

Figure 1. CASE OF FINITE COMPUTATION EXEMPLIFIED
Middle row, (with red arrows) represents computation of $Gr1$, elements k_i and m_i are given by the recurrence (rec2)
third row shows computation of $Gr3$, the subsequent triples are $X_{i+1} = i+1, Y_{i+1} = 3Y_i + 2_i^Z, Z_{i+1} = Z_i + k_i$
first row (blue arrows) shows computation of algorithm IC on triples, $\bar{Y}_x = Y_x$ and $\bar{Z}_x = Z_x$ and for $i = x, \ldots, 1$ we have $\bar{Z}_{i-1} = \bar{Z}_i - k_i$ and $\bar{Y}_{i-1} = (\bar{Y}_i / 2^{k_i}) - 3^{i-1}$

2.6. The Figure 1 , states the same as point 2, see Figure 5 p. 17.

3.1. We need to prove that if for a certain element n its computation is infinite then $n$is non-standard element.

3.2. Let n be such element, the sequence of consecutive values of the variable n is shown in the following diagram

$$n \xrightarrow[/2^{k_0}]{} m_1 \xrightarrow[/2^{k_1}]{3m_1+1} m_2 \xrightarrow[/2^{k_2}]{3m_2+1} m_3 \cdots$$

An enriched version of the algorithm has infinite computation and shows the increasing, infinite sequence of triples

$$
\begin{tikzcd}
n \arrow[r, "/2^{k_0}"'] \arrow[d, equal] & m_1 \arrow[r, "3m_1+1", "/2^{k_1}"'] \arrow[d, equal] & m_2 \arrow[d, equal] & \cdots & \rightarrow m_{x-1} \arrow[r, "3m_{x-1}+1", "/2^{k_x}"'] \arrow[d, equal] & m_x \neq 1 \rightarrow \arrow[d, equal] & \underline{\cdots \rightarrow} \\
\langle 0,0,0\rangle \arrow[r, "\Gamma_3"'] & \langle 0,0,k_0\rangle \arrow[r, "\Delta_3"'] & \langle 1,1,,k_0\rangle & \cdots & \rightarrow \langle x-1, Y_{x-1}, Z_{x-1}\rangle \arrow[r, "\Delta_3"'] & \langle x, Y_x, Z_x\rangle \rightarrow & \underline{\cdots \rightarrow}
\end{tikzcd}
$$

Note, for all $i \in N$ the lower row contains a triple $\langle i, Y_i, Z_i\rangle$. The sequence of these triples is monotone, increasing.

3.3. Therefore any infinite computation of Collatz algorithm contains some unreachable elements.

3.4. Therefore the entire computation consists of non-standard elements.

3.5. Hence, its origin, the element n is a non-standard element.

Historical remarks

Pàl Erdés said on Collatz conjecture: *"Mathematics may not be ready for such problems."* Let's see.
Mojżesz Presburger has proved the completeness and decidability of arithmetic of addition of natural nubers in 1929.
In the same year S. Jaśkowski found a non-standard model of Presburger theory (see, nnote of A. Tarski of 1934) .
Kurt Gödel (1931) published his theorem on incompleteness of Peano arithmetic.
Stephen Kleene shown (in 1936) that any recurrence defining a computable function can be replaced by the operation of effective minimum (nowadays one can say every recursive function in the integers, is programmable by means **while** instruction).

It seems that P. Erdés was wrong, his colleagues Rozsa Peter and Laszlo Kalmar were able to point it out.